13.2 Derivatives and Integrals of Vector Functions

Definition: The **derivative of** \( \mathbf{r} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \)

Note: the vector \( \mathbf{r}'(t) \) is a vector tangent to the curve \( C \) defined by \( \mathbf{r}(t) \) at any point.

Definition: **unit tangent vector** \( \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} \).

Theorem: If \( \mathbf{r}(t) = <f(t), g(t), h(t)> \), then \( \mathbf{r}'(t) = <f'(t), g'(t), h'(t)> \)

example: 2-dimensional (with graph) \( \mathbf{r}(t) = <t - 3, t^2 - 1> \)

example: Helix \( <\cos t, \sin t, t> \)

Definition: the **second derivative** \( \mathbf{r}''(t) = (\mathbf{r}'(t))' \)

Definition: \( \mathbf{r}(t) \) is smooth in an interval \( I \) if \( \mathbf{r}'(t) \) is continuous and \( \mathbf{r}'(t) \neq \mathbf{0} \), except possibly at the endpoints of \( I \).

Theorem: (Differentiation Rules) Suppose \( \mathbf{u} \) and \( \mathbf{v} \) are differentiable vector functions, \( c \) a scalar, and \( f \) a real-valued function: then

1. \( \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \mathbf{u}'(t) + \mathbf{v}'(t) \)
2. \( \frac{d}{dt}(c\mathbf{u}(t)) = c\mathbf{u}'(t) \)
3. \( \frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \)
4. \( \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' \)
5. \( \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}' \)
6. \( \frac{d}{dt}(\mathbf{u}(f(t))) = f'(t)\mathbf{u}'(f(t)) \)

Fact: If \( ||\mathbf{r}(t)|| = c \) (a constant), then \( \mathbf{r}'(t) \) is orthogonal to \( \mathbf{r}(t) \).

Use: \( \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = 0 \) to show that \( \mathbf{r} \cdot \mathbf{r}' = 0 \)
13.2 Derivatives and Integrals of Vector Functions (cont'd)

Integrals:

Definition: The **definite integral** of \( \mathbf{r}(t) \), denoted \( \int_a^b \mathbf{r}(t) \, dt \), is given by

\[
\int_a^b \mathbf{r}(t) \, dt = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*) \Delta t = \lim_{n \to \infty} \left< \sum_{i=1}^{n} f(t_i^*) \Delta t, \sum_{i=1}^{n} g(t_i^*) \Delta t, \sum_{i=1}^{n} h(t_i^*) \Delta t \right>
\]

so \( \int_a^b \mathbf{r}(t) \, dt = \left< \int_a^b f(t) \, dt, \int_a^b g(t) \, dt, \int_a^b h(t) \, dt \right> \)

The Fundamental Theorem of Calculus extends to vector functions: If \( \mathbf{R}(t) \) is any anti-derivative of \( \mathbf{r}(t) \) [i.e. \( \mathbf{R}'(t) = \mathbf{r}(t) \)], then

\[
\int_a^b \mathbf{r}(t) \, dt = \mathbf{R}(t) \bigg|_a^b = \mathbf{R}(b) - \mathbf{R}(a)
\]

Note that with indefinite integrals, \( \int \mathbf{r}(t) \, dt = \mathbf{R}(t) + \mathbf{C} \)

example: indefinite integral

example: 13.2.38 Evaluate \( \int_1^2 \left( t^2 \mathbf{i} + t \sqrt{t - 1} \mathbf{j} + t \sin(\pi t) \mathbf{k} \right) \, dt \)