12.5 Equations of Lines and Planes

Recall: in \( \mathbb{R}^2 \), a line is determined by a point and a 'slope' (i.e. direction).

In \( \mathbb{R}^3 \), the **vector equation** of a line \( \mathbb{L} \) is given by

\[
\mathbf{r} = \mathbf{r}_0 + t \mathbf{v}, \quad \text{where } t \text{ is a parameter.}
\]

**Parametric equations** of a line \( \mathbb{L} \):

\[
\begin{align*}
x &= x_0 + t a \\
y &= y_0 + t b \\
z &= z_0 + t c
\end{align*}
\]

**Symmetric equations** of a line \( \mathbb{L} \):

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]

\( a, b, c \) are called the **direction numbers** of \( \mathbb{L} \).

Example: line through two points

-- intersection with a coordinate plane.

Parallel Lines: same direction vector (but different line)

Skew lines: not parallel, and don't intersect

Example:

\[
\begin{align*}
x &= 4 + 2t & \quad x &= 2 + u \\
y &= -5 + 4t & \quad y &= -1 + 3u \\
z &= 1 - 3t & \quad z &= 2u
\end{align*}
\]

lines are skew

Note: If parametric equation for \( z \) in \( \mathbb{L}_2 \) is changed to \( z = -2u \), the lines will intersect.
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Planes: determined by a point \((x_0, y_0, z_0)\) and a (orthogonal) direction vector \(\mathbf{n} = <a, b, c>\). If \(P = (x, y, z)\) is a point on the plane, equation is:

\[\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\] (called the vector equation of the plane)

yields \(a(x - x_0) + b(y - y_0) + c(z - z_0) = 0\)

\[\rightarrow ax + by + cz + d = 0\]

Example: Find the equation of the plane through \(1, 2, 0\), \(5, 3, 2\), & \(4, -1, 7\).

Example (12.5.12): Find the line of intersection of planes given by the equations

\[x + 2y + 3z = 1\] and \(x - y + z = 1\)

Two planes are parallel if their normal vectors are parallel.

The angle between two planes is the (acute) angle formed by the two normal vectors.

Example: Find the angle between the planes \(4x + 4y - z = 8\) and \(2x - y + 3z = 6\)

Distance from a point \((x_1, y_1, z_1)\) and a plane \(\mathbb{P}: ax + by + cz = d\) is

\[D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}\]

Example (12.5.71): Find the distance from the point \((1, -2, 4)\) to the plane

\[3x + 2y + 6z = 5\]