Math 22 - Calculus of Several Variables Midterm Exam
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1. Given: \( \mathbf{u} = < 8, -2, 6 >, \quad \mathbf{v} = < 3, -4, -1 >, \quad \mathbf{w} = < 1, 3, 1 > \)
   Find:
   a. \( \mathbf{u} \cdot (2\mathbf{v} - 5\mathbf{w}) \)
   b. a unit vector in the opposite direction of \( \mathbf{w} \).
   c. \( \mathbf{v} \times \mathbf{w} \)
   d. the angle formed by \( \mathbf{u} \) and \( \mathbf{v} \).
   e. \( \text{proj}_w(\mathbf{u}) \) [the vector projection of \( \mathbf{u} \) onto \( \mathbf{w} \)]

2. a. Find the parametric equations of the line \( \mathbb{L} \) passing through the points \( (4, -1, 5) \) and \( (6, 2, 1) \).
   b. Find the point at which the line \( \mathbb{L} \) passes through the plane whose equation is \( x + 2y + 3z = 5 \).
   c. The line \( \mathbb{L} \) intersects the line given by the vector equation \( \mathbf{r} = < 5, -4, 3 > + s < 1, 2, -2 > \). Find the point of intersection.

3. Given the points \( (5, 0, 1), (0, -1, 2), \) and \( (8, -1, 1) \), find the equation of the plane that passes through these points. Write your answer in the form \( ax + by + cz = d \).

4. Evaluate: \( \int_0^2 \left( e^{3t} \mathbf{i} + t e^t \mathbf{j} - t e^t \mathbf{k} \right) dt \)

5. Given the vector function \( \mathbf{r}(t) = < -\sin t, \sqrt{3} \sin t, 2 \cos t > \)
   a. Find the length of the curve from \( t = 0 \) to \( t = \pi \).
   b. Find the unit tangent vector \( \mathbf{T} \) and the unit normal vector \( \mathbf{N} \). [Note that you will need to find the value of \( t \) that yields this point].

6. Given \( f(x, y) = \tan(3y^2 - x) + \ln(x^2 + 7y) \), find:
   a. \( f_y(x, y) \)
   b. \( f_{xx}(x, y) \)
   c. \( f_{xy}(x, y) \)

7. Given the function \( z = x^2 \sin(y) \) where \( x = s \cdot \cos t \) and \( y = \frac{t}{s} \)
   a. Use the chain rule to find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \) : you can write your answer in terms of any combination of variables.
   b. Evaluate each partial derivative in part a when \( s = 3 \) and \( t = \pi \).
1. a. 12  
   b. \( < -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}, -\frac{1}{\sqrt{11}} > \)
   c. \( < -1, -4, 13 > \)  
   d. \( \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \)  
   e. \( < \frac{8}{11}, \frac{24}{11}, \frac{8}{11} > \)

2. a. \( x = 4 + 2t \)  
   b. \( (10, 8, -7) \)  
   c. \( (14, 14, -15) \)  
   [can also 'start' with other point]

3. \( x + 3y + 8z = 13 \)

4. \( < \frac{e^{6} - 1}{3}, e^2 + 1, \frac{1-e^4}{2} > \)

5. a. \( \int_{0}^{\pi} \sqrt{(-\cos t)^2 + \left(\sqrt{3}\cos t\right)^2 + (-2\sin t)^2} \, dt = \int_{0}^{\pi} \sqrt{4} \, dt = (2t) \bigg|_{0}^{\pi} = 2\pi \)
   b. \( \mathbf{T} = < -\cos \frac{t}{2}, \sqrt{3}\cos t, -\sin t > \)  
   \( \mathbf{N} = < \sin t \frac{1}{2}, -\sqrt{3}\sin t, -\cos t > \)

6. a. \( 6y\sec^2(3y^2 - x) + \frac{7}{x^2 + 7y} \)
   b. \( 2\sec^2(3y^2 - x)\tan(3y^2 - x) + \frac{14y - 2x^2}{(x^2 + 7y)^2} \)
   c. \( -12y\sec^2(3y^2 - x)\tan(3y^2 - x) - \frac{14x}{(x^2 + 7y)^2} \)

7. a. \( \frac{\partial z}{\partial s} = (2x\sin y)(\cos t) + (x^2\cos y)\left(-\frac{t}{s^2}\right) \)
   \( \frac{\partial z}{\partial t} = (2x\sin y)(-s\sin t) + (x^2\cos y)\left(\frac{1}{s}\right) \)
   b. \( \frac{\partial z}{\partial s} \bigg|_{(s,t)=(3,\pi)} = 3\sqrt{3} - \frac{\pi}{2} \)  
   \( \frac{\partial z}{\partial t} \bigg|_{(s,t)=(3,\pi)} = \frac{3}{2} \)