8.1 Arc Length and Surface Area

Finding the length of a function \( y = f(x) \) between two points. Calculate the length of a curve by adding the estimates of small pieces of the curve.

\[
\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x
\]

\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx
\]

example: (difficult) Find the length of the curve \( y = x^2 \) from \((0, 0)\) to \((3, 9)\). Set up only.

example: Find the length of the curve \( y = x^{3/2} \) from \((0, 0)\) to \((4, 8)\).

example (8.1.4): Find the length of the curve given by the equation

\[
y = \left(\frac{x}{2}\right)^4 + \frac{1}{2x^2} \quad \text{over } [1, 4].
\]

example: Find the length of the curve \( y = \frac{2}{3}(x^2 - 1)^{3/2} \) for \(1 \leq x \leq 3\).

Surface area of revolution. Calculate the area of a surface of revolution by calculating the area of 'strips' generated by rotating a 'chunk' of arclength \( \Delta s \) around the axis of rotation, and taking a Riemann sum.

\[
A_{\text{strip}} \approx (\text{circumference})(\Delta s) = 2\pi(R) \Delta s
\]

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi(R) \, ds = \int_{a}^{b} 2\pi(R) \sqrt{(dx)^2 + (dy)^2}
\]

Note: there are two integral formulas integrating with respect to \( x \), depending on which axis is the axis of rotation.

Rotating around the \( x \)-axis yields \( R = f(x) \), so

\[
A = \int_{a}^{b} 2\pi(f(x)) \sqrt{1 + (f'(x))^2} \, dx
\]
8.1 Arc Length and Surface Area (continued)

Rotating around the \( y \)-axis yields \( R = x \), so
\[
A = \int_a^b 2\pi(x) \sqrt{1 + (f'(x))^2} \, dx
\]

example: Find the area of the surface of revolution obtained by rotating the curve
\[
y = \frac{x^3}{6} + \frac{1}{2x}
\]
for \( \frac{1}{2} \leq x \leq 1 \) about the \( x \)-axis.

example: Find the area of the surface of revolution obtained by rotating the curve \( y = x^2 \)
from \((0, 0)\) to \((3, 9)\) around the \( y \)-axis.

example: Find the area of the surface of revolution obtained by rotating the curve \( y = \sqrt[3]{x} \)
from \((1, 1)\) to \((8, 2)\) around the \( y \)-axis.

example: Calculate the surface area of a sphere of radius \( A \).