Chapter 6 - Applications of the Integral

6.1 Area Between Two Curves

Area bounded by two curves \( y = f(x) \) and \( y = g(x) \) is:

\[
A(x) = \int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx
\]

[Here the region is \textbf{vertically simple}, i.e. \( f(x) \geq g(x) \) on \( [a, b] \) ]

example: \( y = e^x , \, y = \sqrt{x} \) from \( x = 0 \) to \( x = 4 \).

example: \( y = x^2 , \, y = x + 2 \) (Note: find points of intersection)

Boundaries may change for an area, so more than one integral may be needed to calculate the area.

example (6.1.19): Find the area of the region bounded by the curve \( y = \cos x \), the line through the origin and the point \( \left( \frac{\pi}{6}, \frac{\sqrt{3}}{2} \right) \) and the line through the origin and the point \( \left( \frac{\pi}{3}, \frac{1}{2} \right) \).

If \( f(x) \) and \( g(x) \) cross several times in the interval \( [a, b] \), the area is

\[
A(x) = \int_a^b \left| f(x) - g(x) \right| \, dx
\]

example: Find the area bounded by \( y = \sin x \) and \( y = \cos x \) on the interval \( [0, \frac{3\pi}{4}] \).

Horizontal strips can be used if the region has boundaries that can be described using functions of \( x \): in this case the 'width' = \( dy \), so area is \( \int_c^d \left| x_R - x_L \right| \, dy \)

example:

\[
x + y = 6
\]

\[
x - y^2 = 0
\]