10.5 The Ratio and Root Tests and Strategies for Choosing Tests

Theorem 1 (Ratio Test): Given the series \( \sum_{k=1}^{\infty} a_k \), consider \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho : \)

1) if \( \rho < 1 \), the the series \( \sum_{k=1}^{\infty} a_k \) is absolutely convergent.

2) if \( \rho > 1 \), the the series \( \sum_{k=1}^{\infty} a_k \) is divergent.

3) if \( \rho = 1 \), then the test is inconclusive (look at \( \sum \frac{1}{n^2} \) and \( \sum \frac{1}{n} \))

examples: \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{2^n} \) \( \sum_{n=1}^{\infty} \frac{n!}{10^n} \) \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \)

Theorem 1 (Root Test): Given the series \( \sum_{k=1}^{\infty} a_k \), consider \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L : \)

1) if \( L < 1 \), the the series \( \sum_{k=1}^{\infty} a_k \) is absolutely convergent.

2) if \( L > 1 \), the the series \( \sum_{k=1}^{\infty} a_k \) is divergent.

3) if \( L = 1 \), then the test is inconclusive.

examples: \( \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n} \) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan n)^n} \)

Note: a conditionally convergent series can converge to any number; one only needs to rearrange the terms.
10.5 The Ratio and Root Tests and Strategies for Choosing Tests (continued)

example: alternating harmonic series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \)

(valid since \( 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \) diverges to \( \infty \)

and \(- \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \ldots \) diverges to \(- \infty \))

Which test should be used?

At the start ask: is it a \( p \)-series? is it a geometric series?

1) if \( \lim_{n \to \infty} a_n \neq 0 \), then the \( n^{th} \) term divergence test indicates the series diverges.

2) If all the terms are positive, try:
   a) direct comparison test: making numerators smaller and/or denominators larger makes the term smaller, while making numerators larger and/or denominators smaller makes the term larger.
   b) limit comparison test: good with algebraic terms. Consider the ratio of dominant terms in the numerator and denominator.
   c) ratio test: especially useful with terms that contain factorials and constants raised to the \( n^{th} \) power.
   d) root test: if the terms of the series are of the form \( f(n)^{g(n)} \).
   e) integral test: if the terms \( a_n = f(n) \), where \( \int_{1}^{\infty} f(x) \, dx \) is easily evaluated.

3) Series that are not positive series
   a) alternating series test: use if the series is of the form \( \sum_{n=1}^{\infty} (-1)^{n-1} a_n \).
   b) absolute convergence: if terms are randomly positive and/or negative, check for absolute convergence.
10.5 The Ratio and Root Tests and Strategies for Choosing Tests (continued)

examples: \[ \sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}} \quad \sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1} \]

\[ \sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}} \quad 10.5.48 \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \]