10.4 Absolute and Conditional Convergence

Definition: A series \( \sum_{n=1}^{\infty} a_n \) is **converges absolutely** if the series of absolute values \( \sum_{n=1}^{\infty} |a_n| \) is convergent.

example: \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \): absolutely convergent

Theorem 1: If a series is absolutely convergent, then it converges.

Definition: A series \( \sum_{n=1}^{\infty} a_n \) is **conditionally convergent** if the series converges, but is not absolutely convergent.

example: \( \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n} \): conditionally convergent

An **alternating series** is a series whose terms are alternatively positive and negative.

example: \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \)

example: \( \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n-1} \)

Alternating series are of form \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) or \( \sum_{n=1}^{\infty} (-1)^n b_n \) where \( b_n \) is a positive number.
10.4 Absolute and Conditional Convergence (continued)

Theorem 2 (alternating series test): If the series \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) satisfies:

1) \( b_{n+1} \leq b_n \) for all \( n \) and 2) \( \lim_{n \to \infty} b_n = 0 \), then the series \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) is convergent. Furthermore, \( 0 < S < b_1 \) and \( S_{2N} < S < S_{2N+1} \).

Proof: the even partial sums \( s_n \) form an increasing sequence:

\[
S_{2n} = (b_1 - b_2) + (b_3 - b_4) + \ldots + (b_{2n-1} - b_{2n})
\]

which is bounded above (by \( b_1 \));

\[
S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \ldots - (b_{2n-2} - b_{2n-1}) - b_{2n}
\]

hence a limit exists: \( \lim_{n \to \infty} S_{2n} = S \).

The odd partial sums are \( S_{2n+1} = S_{2n} + b_{2n+1} \); so \( \lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} (S_{2n} + b_{2n+1}) = S + 0 = S \).

example: \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \) (the alternating harmonic series) converges.

Theorem 3: Let \( S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n \) where \( \{b_n\} \) is a positive decreasing sequence that converges to 0. Then \( |S - S_N| < b_{N+1} \).