3 Final Review Sessions; Check Website for Times/Places.

Eastman's Office Hours During Finals Week: Monday, 18 March Noon-3pm.

10.7 Taylor Series (Cont'd)

Given \( f(x) \): If \( f \) is infinitely differentiable then the Taylor Series for \( f \) centered at \( x = c \) is:

\[
T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n
\]

\( \exp f(x) = \sin x \) Find the Taylor Series for \( f(x) \) about \( x = 0 \). (So we are finding the Maclaurin Series for \( f(x) \)
\[
\begin{align*}
T(x) &= 1 + \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \\
&= 1 + \sum_{k=0}^{\infty} \frac{(-1)^k 2k+1}{(2k+1)!} x^{2k+1}
\end{align*}
\]

**Interval of Convergence**

**Ratio Test**

\[
1 + \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = 1 + \lim_{k \to \infty} \left| \frac{\frac{(-1)^{k+1} 2k+3}{(2k+3)!}}{\frac{(-1)^k 2k+1}{(2k+1)!}} \right|
\]
\[
\left. \lim_{k \to \infty} \frac{x^2}{(2k+2)(2k+3)} \right| = 0 < 1 \text{ for all } x > 0
\]

Valid for all \( x \) \((R = \infty)\)

So \( \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \)

\[e^x = \frac{1}{x} \quad \text{Find Taylor Series centered at } x = 1\]

\[
\begin{align*}
\frac{f(x)}{x^1} &= -1 \\
\frac{f'(x)}{x^2} &= -2 \\
\frac{f''(x)}{x^3} &= -3 \\
\frac{f'''(x)}{x^4} &= -4 \\
\frac{f^{(4)}(x)}{x^5} &= -5 \\
\frac{f^{(5)}(x)}{x^6} &= -6 \\
\frac{f^{(6)}(x)}{x^7} &= -7 \\
\frac{f^{(7)}(x)}{x^8} &= -8 \\
\frac{f^{(8)}(x)}{x^9} &= -9 \\
\frac{f^{(9)}(x)}{x^{10}} &= -10 \\
\end{align*}
\]

\[
\begin{align*}
&= 1 \quad \rightarrow \quad a_0 = 1 \\
&= -1 \quad \rightarrow \quad a_1 = \frac{-1}{1!} = -1 \\
&= 2 \quad \rightarrow \quad a_2 = \frac{2}{2!} = 1 \\
&= -6 \quad \rightarrow \quad a_3 = \frac{-6}{3!} = -1 \\
&= 24 \quad \rightarrow \quad a_4 = \frac{24}{4!} = 1 \\
&= -120 \quad \rightarrow \quad a_5 = -1 \\
\end{align*}
\]

\[
f^{(n)}(x) = (-1)^n \frac{n!}{x^n}
\]

\[
T(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{n!}
\]
INTERVAL OF CONVERGENCE?

**RATIO TEST**

\[
\lim_{n \to \infty} \left| \frac{(x-1)^{n+1}}{(x-1)^n} \right| = \lim_{n \to \infty} \left| x-1 \right| = \left| x-1 \right|
\]

**CONV** IF \( \left| x-1 \right| < 1 \)

\[-1 < x-1 < 1 \]

\[-1 + 1 < 0 \]

\[0 < x < 2 \]

\[x=2 : \ T(2) = \sum_{n=0}^{\infty} (-1)^n ([2]-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \ldots \]

\[x=0 : \ T(0) = \sum_{n=0}^{\infty} (-1)^n ([0]-1)^n = \sum_{n=0}^{\infty} (-1)^n = \sum_{n=0}^{\infty} 1 \]

\[= 1 + 1 + 1 + 1 + \ldots \]

\[\therefore \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \text{ For } x \in (0,2) \]

**DEN THE nth TAYLOR POLYNOMIAL**

\[T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k \]

**DEN THE (n th) REMAINDER OF A TAYLOR SERIES**

\[R_n(x) = f(x) - T_n(x) \text{ [ } R_n \to 0 \text{ in conv } \]
Find the Taylor series for:

2) \( f(x) = x^2 \sin x \) (can multiply)

\[
= x^2 \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \cdot x^2
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}
\]

TRUE \( \forall x \)

b) \( g(x) = \tan x = \frac{\sin x}{\cos x} \)

\[
= \frac{x^3}{6} - \frac{x^5}{15} + \frac{x^7}{42} - \frac{x^9}{315} + \ldots
\]

\( g'(x) = \sec^2 x \)

\( g''(x) \leftarrow \text{CHAIN RULE} \)

\[
x + \frac{x^3}{3}
\]

\[
1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots
\]

\[
- \frac{x^3}{6} + \frac{x^3}{2 \cdot 3} = \frac{2x^3}{6}
\]

Calculus w/ Power Series,

Evaluate \( \int_0^1 e^{-x^2} \) dx
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]

\[ e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots \]

\[ \int_0^1 e^{-x^2} \, dx = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \left|_0^1 \right. \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n! (2n+1)} \bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n (1)}{n! (2n+1)} \]

\[ = x - \frac{x^3}{3} + \frac{x^5}{3! 5} - \frac{x^7}{3! 7} + \ldots \bigg|_0^1 = 1 - \frac{1}{3} + \frac{1}{21.5} - \frac{1}{31.7} + \frac{1}{41.9} - \ldots \]

Using the alternating series test, so \[ |f(x) - T_n(x)| < \frac{1}{41.9} \]

Error \[ < \left( \frac{1}{5!} \right) = \frac{1}{120.11} = \frac{1}{1320} \]
FINAL EXAM - 10 PROBLEMS/10 PAGES
15-20 RESPONSES

3 PROBLEMS \rightarrow INTEGRATION TECHNIQUES

1 PROBLEM \rightarrow IMPROPER INTEGRALS

3 PROBLEMS \rightarrow APPLICATIONS

NO WORK

NO FLUID PRESSURE

2 PROBLEMS \rightarrow SERIES

\sum_{n=0}^{\infty} \frac{(4)^n}{n!} = e^{(4)}

2 PROBLEMS \rightarrow TAYLOR POLY/SERIES

FIND A TAYLOR POLY. / FIND A SERIES, THEN APPROX AN INTEGRAL.