4.8 Newton's Method

Used in estimating roots of equations.

Note that slope of tangent line at \( x_0 \) (\( x_0 \) is the starting value called the intial guess).
\[ m_{tan} = f'(x_0) \]. It crosses the \( x \)-axis at \( (x_1, 0) \), so slope of the tangent line is also
\[ m_{tan} = \frac{-f(x_0)}{x_1-x_0} \]. Solving for \( x_1 \) yields:
\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \]. Repeat this procedure with the point \( (x_1, f(x_1)) \).
In general, \[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]. If all goes well, we have \( \lim_{n \to \infty} x_n = r \).

Newton's Method: To approximate the root of \( f(x) = 0 \):

Step 1: Choose an initial guess \( x_0 \) (close to the desired root if possible).
Step 2: Generate successive approximations \( x_0, x_1, x_2, \ldots \), where
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

example: Find an approximation for \( \sqrt{5} \). Let \( f(x) = x^2 - 5 \)
\[ x_{n+1} = x_n - \frac{(x_n^2-5)}{2x_n} \]

example: Find the solution to the equation \( \cos x = x \)

Note that the function \( f(x) = x - \cos x \) will have a root at the solution of the equation.
Note: If Newton's method yields a sequence that diverges, try choosing a better initial value.