3.9 Derivatives of General Exponential and Logarithmic Functions

Theorem 1: \( \frac{d}{dx} (\ln x) = \frac{1}{x} \)

Proof: Use implicit differentiation. If \( y = \ln x \) then \( e^y = e^{\ln x} = x \).

\[
\frac{d}{dx} (e^y) = \frac{d}{dx} (x) \Rightarrow e^y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}
\]

Examples: Differentiate: a) \( f(x) = x^2 \ln x \)  b) \( g(x) = (\ln x)^3 \)

Note: \( \frac{d}{dx} (\log_b x) = \frac{1}{x \ln b} \) [using the change of base formula]

Recall: \( \frac{d}{dx} (b^x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x b^h - b^x}{h} = \lim_{h \to 0} (b^x) \frac{b^h - 1}{h} \)

Theorem 2: \( \frac{d}{dx} (b^x) = (\ln b) \ b^x \) for \( b > 0 \)

Proof: \( \frac{d}{dx} (b^x) = \frac{d}{dx} (e^{\ln b \cdot x}) = (e^{\ln b \cdot x})'[\ln b \cdot x]' = b^x \cdot \ln b \)

Alternate proof: If \( y = b^x \), we want to find \( \frac{dy}{dx} \). Take natural logs of both sides

\( \ln (y) = \ln (b^x) = x \ln b \), then implicitly differentiate both sides and solve for \( \frac{dy}{dx} \).

Examples: Differentiate: a) \( f(x) = 3^{2x-1} \)  b) \( g(x) = 7^{\tan x} \)

General Formulas: \( \frac{d}{dx} (\ln u) = \frac{1}{u} \frac{du}{dx} \) \( \frac{d}{dx} (\ln |f(x)|) = \frac{f'(x)}{f(x)} \)

Examples: \( \frac{d}{dx} (\ln (\sec x)) \) \( \frac{d}{dx} (\sin(\ln x + \frac{\pi}{3})) \)
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Logarithmic differentiation: use when \( y = (g(x))^{h(x)} \)

1) Take natural logarithms of both sides of the equation; use the laws of logarithms to simplify.
2) Differentiate implicitly with respect to \( x \).
3) Solve the resulting equation for \( y' = \frac{dy}{dx} \) (subbing for \( y \))

example: \( y = (x^2 + 1)^x \)  
example: \( y = (\sin x)^{(x+3)} \)

Hyperbolic Functions

Definition: hyperbolic sine function: \( \sinh x = \frac{e^x - e^{-x}}{2} \), and the hyperbolic cosine function: \( \cosh x = \frac{e^x + e^{-x}}{2} \) [Note that these functions are discussed in Chapter 1].

Four other hyperbolic functions are defined similarly to trigonometric functions. The derivatives of these hyperbolic functions are as follows:

\[
\frac{d}{dx} (\sinh x) = \cosh x \quad \frac{d}{dx} (\cosh x) = \sinh x
\]

\[
\frac{d}{dx} (\tanh x) = \text{sech}^2 x \quad \frac{d}{dx} (\coth x) = - \text{csch}^2 x
\]

\[
\frac{d}{dx} (\text{sech} x) = - \text{sech} x \tanh x \quad \frac{d}{dx} (\text{csch} x) = - \text{csch} x \coth x
\]

examples: Verify that \( \frac{d}{dx} (\tanh x) = \text{sech}^2 x \)

example: Differentiate: a) \( f(x) = \sinh(4x^3 + 1) \)  
b) \( g(x) = \sqrt{\cosh(3x) + 4} \)

Note: The domains and the derivatives of the inverse hyperbolic functions are listed in the text at the end of the section (p. 179). These derivatives may appear in the homework problems but they will not be on the midterm nor the final exams.