3.3 The Product and Quotient Rules

Note: \([f \cdot g]' \neq f' \cdot g'\)

Theorem 1 (Product Rule): If \(f\) and \(g\) are both differentiable, then

\[
\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx} (g(x)) + \frac{d}{dx} (f(x)) \cdot g(x)
\]

Using prime notation:

\[
(fg)'(x) = f(x) g'(x) + f'(x) g(x)
\]

Proof:

\[
\frac{d}{dx} (f(x) \cdot g(x)) = \lim_{h \to 0} \frac{f(x+h) g(x+h) - f(x) g(x)}{h} = \\
= \lim_{h \to 0} \frac{f(x+h) g(x+h) + [-f(x+h) g(x)+f(x+h) g(x)] - f(x) g(x)}{h} = \\
\]

example: Find the derivative of \(F(x) = x^2(7x - 5)\)

example: Find the derivative of \(F(x) = \sqrt{x}(4 - x^3)\)

example: Find the derivative of \(F(x) = e^x(x^4 - 8x + 3)\)

Theorem 2 (Quotient Rule): If \(f\) and \(g\) are both differentiable, then

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \cdot \frac{d}{dx} (f(x)) - f(x) \cdot \frac{d}{dx} (g(x))}{(g(x))^2}
\]

Using prime notation:

\[
\left( \frac{f}{g} \right)'(x) = \frac{g(x) f'(x) - f(x) g'(x)}{(g(x))^2}
\]

example: Find the derivative of \(G(x) = \frac{x^3+5}{x+e^x}\)

example: Find the equation of the line tangent to the graph of \(f(x) = \frac{x}{x^2+1}\) at the point \((3, \frac{3}{10})\).