3.2 The Derivative as a Function

Definition: The **derivative function** is denoted by \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

The **domain** of \( f'(x) \) is the set of values \( x \) where the limit exists.

Example: Find \( f'(x) \):

a) \( f(x) = x^3 + 5x \)  

b) \( f(x) = \sqrt{x + 4} \)  

c) \( f(x) = \frac{1}{x^2} \)

Notation: \( f'(x) = \frac{dy}{dx} = y' = \frac{df}{dx} = \frac{d}{dx}(f(x)) = Df(x) \)

Definition: A function \( f \) is **differentiable at** \( x = a \) if \( f'(a) \) exists.

Theorem 1: For all exponents \( n \), \( \frac{d}{dx}(x^n) = n \cdot x^{n-1} \)

Proof uses Binomial Theorem to expand \( (x + h)^n \) [for whole numbers \( n \)]

Proof: \( \frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \)

Example: \( \frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{1}{3} \cdot x^{\left(\frac{1}{3}\right)-1} = \frac{1}{3x^{\frac{2}{3}}} \)

Theorem 2: If \( f \) and \( g \) are both differentiable, and if \( c \) is a constant, then

\[
\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x) \\
\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x) \\
\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)
\]

Exponential Functions:

Theorem 3: \( e \) is the number such that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \)

Justification is a graphical argument. Consider \( \frac{d}{dx}(b^x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} \)

\[
= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}
\]
3.2 The Derivative as a Function (p. 2)

\[ b = 2 : \begin{array}{|c|c|c|c|c|c|} \hline h & .1 & .01 & .001 & .0001 & \text{--} .0001 \\
\hline \frac{2^h - 1}{h} & & & & & \\
\hline \end{array} \]

Guess: \( \frac{d}{dx}(2^x) \approx 2^x( ) \)

\[ b = 3 : \begin{array}{|c|c|c|c|c|c|} \hline h & .1 & .01 & .001 & .0001 & \text{--} .0001 \\
\hline \frac{3^h - 1}{h} & & & & & \\
\hline \end{array} \]

Guess: \( \frac{d}{dx}(3^x) \approx 3^x( ) \)

Theorem 4: If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \)

Note the converse is not true!

example: Given \( f(x) = |x| \), for what value(s) of \( x \) is \( f \) differentiable?

Where is a function \( f \) not differentiable?

-- at corners/sharp turns

-- at discontinuities (any type)

-- at points where the tangent line is vertical