2.4 Limits and Continuity

Definition: If a function \( f(x) \) is defined on an open interval containing \( x = c \). Then \( f \) is **continuous at \( x = c \)** if \( \lim_{x \to c} f(x) = f(c) \).

Three components of this statement:

1) \( f(c) \) exists (\( c \) is in the domain of \( f \))
2) \( \lim_{x \to c} f(x) \) exists
3) \( \lim_{x \to c} f(x) = f(c) \)

Note: If the function is not continuous at \( x = c \), we say that \( f \) is **discontinuous** at \( x = c \).

If a function \( f(x) \) is continuous at all points in an interval \( I \), then we say that \( f(x) \) is continuous on \( I \). If the interval \( I = [a, b] \), then we need (for continuity) \( \lim_{x \to a^+} f(x) = f(b) \) and \( \lim_{x \to b^-} f(x) = f(b) \).

Examples:

- Graphs
  
  \[
  f(x) = \begin{cases} 
    \frac{x^2 - 4}{x-2} & x \neq 2 \\
    5 & x = 2 
  \end{cases}
  \]

- \[
  g(x) = \frac{1}{(x-3)^2} 
  \] (and 4 at \( x = 3 \))

- \[
  f(x) = \lfloor x \rfloor 
  \] (the greatest integer function)

Types of discontinuities: removable jump infinite

Definition: A function \( f \) is **left-continuous at \( c \)** if \( \lim_{x \to c^-} f(x) = f(c) \).

[Note: **right-continuity** is defined similarly]
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examples:  \[ f(x) = \begin{cases} \frac{x^2+3x}{x+4} & x \leq 2 \\ \frac{3x+4}{x > 2} \end{cases} \]

find the value of \( a \) such that \( f \) is continuous at \( x = 4 \).

Theorem 1: If \( f \) and \( g \) are continuous at \( x = c \), then the following functions are continuous at \( x = c \):

i) \( f(x) + g(x) \) and \( f(x) - g(x) \)  ii) \( k \cdot f(x) \) for and constant \( k \)

iii) \( f(x) \cdot g(x) \)  iv) \( \frac{f(x)}{g(x)} \) (if \( g(c) \neq 0 \))

Theorem 2: Let \( P(x) \) and \( Q(x) \) be polynomials. Then

i) \( P(x) \) is continuous on the real real line (for all real numbers)

ii) \( \frac{P(x)}{Q(x)} \) is continuous on its domain (wherever it is defined)

Theorem 3: The following basic functions are continuous where they are defined:

i) \( y = x^{1/n} \)  ii) \( y = \sin x \) and \( y = \cos x \)

iii) \( y = b^x \) (\( b \neq 1 \))  iv) \( y = \log_b(x) \) (\( b > 0, b \neq 1 \))

Theorem 4: If \( f(x) \) is continuous on an interval \( I \) with range \( R \), and if \( f^{-1}(x) \) exists, then \( f^{-1}(x) \) is continuous with domain \( R \)

Theorem 5: If \( g \) is continuous at \( x = c \), and \( f \) is continuous at \( x = g(c) \), then the composite function \( F(x) = f(g(x)) \) is continuous at \( s = c \).

example:  \[ \lim_{x \to 1} \cos^{-1}\left( \frac{1 - \sqrt{x}}{1 - x} \right) \]