MIDTERM EXAMS WILL BE RETURNED IN SECTION.

3.10 RELATED RATES

If a quantity $Q$ depends on another quantity $x$ [i.e. $Q = f(x)$], what happens if $x$ changes? How does $Q$ change?

\[
\frac{dQ}{dt} = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dx}{dt}
\]

Inflate a spherical balloon.

Pumping in 5 cubic centimeters of air each second:

\[
5 \text{ cu.cm/sec} = \frac{dV}{dt} \quad \text{Known}
\]

How fast is the radius changing when the radius is 4 cm? Find \( \frac{dr}{dt} = ( \ ) \text{ cm/sec} \)
3.10 Related Rates

When one quantity \( Q \) depends on another quantity \( x \), what happens when \( x \) changes? How does \( Q \) change?

example: inflating a (spherical) balloon.

Strategy:

1) Read the problem; draw a diagram if possible.
2) Assign symbols to all quantities that change over time.
3) Express known rates of change in terms of derivatives (w.r.t. \( t \)), and determine which rate(s) is/are asked for.
4) Write an equation that relates the variables whose rates are known and asked for.
5) Implicitly differentiate both sides of this equation (w.r.t. \( t \)).
6) Substitute the (known) values of the variables and the rates into the resultant equation, and solve for the unknown rate.

example: Car A leaves town at noon, heading north at 40 mph. At 1 pm, car B leaves town traveling east at 60 mph. How fast is the distance between the two cars changing at 2 pm?

example: A trough is 10' long and has cross sections in the shape of an isosceles triangle that are 3 feet across at the top and are 2 feet high. If the trough is being filled at a rate of 12 \( \text{ft}^3/\text{min} \), how fast is the water level rising when the water is \( 10' \) deep?

example: A kite 100 feet above the ground moves horizontally at a speed of 8 feet/sec. Find the rate at which the angle between the string and the (horizontal) ground is changing when 200 feet of string have been let out.

examples (3.10.31-32): The pressure \( P \) and the volume \( V \) of an expanding gas are related by the equation \( PV^b = C \) where \( b \) and \( C \) are constants.

31. Find \( \frac{dP}{dt} \) if \( b = 1.2 \), \( P = 8 \text{ kPa} \), \( V = 100 \text{ cm}^2 \), and \( \frac{dV}{dt} = 20 \text{ cm}^3/\text{min} \).

32. Find \( b \) if \( P = 25 \text{ kPa} \), \( \frac{dP}{dt} = 12 \text{ kPa/min} \), \( V = 100 \text{ cm}^3 \), and \( \frac{dV}{dt} = 20 \text{ cm}^3/\text{min} \).
Know \( \frac{dv}{dt} \), find \( \frac{dr}{dt} \)

1. Find relation between \( \sqrt[3]{r} \)

\[
V_{\text{sphere}} = \frac{4}{3} \pi r^3
\]

Want relation between \( \frac{dv}{dt} \) and \( \frac{dr}{dt} \)

\[
d\left( \frac{4}{3} \pi r^3 \right)
\]

\[
\frac{d(v)}{dt} = \frac{d}{dt} \left( \frac{4}{3} \pi r^3 \right)
\]

\[
4 \frac{dv}{dt} = \frac{4}{3} \pi \left[ 3r^2 \frac{dr}{dt} \right]
\]

\[
\frac{4}{4 \pi r^2} \frac{dv}{dt} = \frac{dr}{dt} = \frac{5 \text{ cm}^3}{4 \pi (4\text{ cm})^2} = \frac{5}{64 \pi \text{ cm/s ec}}
\]

\[
\approx 0.025 \text{ cm/sec}
\]

Want to measure rate of change of surface area \( A \): note \( A = 4 \pi r^2 \)

\[
\frac{d(A)}{dt} = \frac{d}{dt} (4\pi r^2) = 8\pi r \frac{dr}{dt}
\]
$$\frac{dA}{dt} = 8 \times 4 \left[ \frac{5}{64 + \pi^2} \right] = \frac{5}{2} \left( \text{cm}^2 \right) / \text{sec.}$$

**EX**

**Cara A**
Leaves at noon
40 mph.

**Cara B**
Leaves at 1 pm,
60 mph.

**Known:**
\[ \frac{dq}{dt} = 40 \text{ miles/hour} \]
\[ \frac{dc}{dt} = 60 \text{ miles/hour} \]

**Find:**
\[ \frac{dc}{dt} \text{ at } 2 \pm \]

\[ a^2 + b^2 = c^2 \]
\[ \frac{d()}{dt} = 2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt} = c \cdot \frac{dc}{dt} \]

\[ \frac{80[40] + 60[60]}{100} = \frac{6400}{10000} = 68 \text{ mph} = \frac{dc}{dt} \]

**How fast is the distance between Cara B and A changing at 2 pm?**

At 2 pm:
\[ \sqrt{80^2 + 60^2} = 100 \]
Find how fast water is rising when depth is 50 cm. (Also when depth is 1.5 m)

\[ \text{Find } V \text{ of water in terms of } h \text{ only.} \]

\[ \text{Area.} \]

\[ V = \text{Area} \times 10 \]

\[ \frac{3}{2} h = b \]

\[ \Rightarrow \frac{3}{2} h = b \]

\[ S_{O} \quad V = 5bh = 5\left(\frac{3}{2}h\right)h = \frac{15}{2}h^2 \]
\[ V = \frac{15}{2} h^2 \]

\[ \frac{d(V)}{dt} = \frac{d}{dt} \left( \frac{15}{2} h^2 \right) \rightarrow \frac{dV}{dt} = \frac{15}{2} \cdot 2h \frac{dh}{dt} \]

\[ \frac{dV}{dt} = \frac{dh}{dt} = \frac{12}{15(0.5)} = \frac{12}{7.5} = 1.6 \text{ m/min} \]

\[ \frac{dh}{dt} = \frac{24}{15} \text{ m/min} \]

\[ = 1.6 \text{ m/min} \]

b) \[ @ h=1.5 \text{ m}, \quad \frac{dh}{dt} = \frac{dV}{dt} \frac{[12]}{15(h)} = \frac{12}{15(1.5)} \]

\[ \frac{dh}{dt} = \frac{12}{4.5} = \frac{24}{4.5} = \frac{8}{15} \text{ m/min} \]