7.3 Direction Fields and Euler's Method

Direction Fields

Given a non-autonomous differential equation \( \frac{dy}{dt} = y' = F(t, y) \) [a function of \( t \) and \( y \)], draw short line segments with slope \( F(t, y) \) at various points \((t, y)\). The resulting graph is called a direction field (or slope field) for the differential equation \( y' = F(t, y) \).

Example: Draw the direction field for \( y' = t + y \).

Once a direction field is drawn, try to visualize a curve that follows the indicated slopes and goes through a given point; this is called a solution curve. For example, we can draw the solution curves in the above example through \((0, 1)\) and \((-2, 0)\).

Example (Example 2 in text): Given the differential equation \( \frac{dp}{dt} = p(1 - p)(2 - 5p) \)

a. Draw the direction field for \(0 \leq t \leq 8\) and \(0 \leq p \leq 1\).

b. Identify all equilibria on the plot.

c. Sketch the solution curve through the point \((0, 0.8)\).

d. What happens to the solution curve as \( t \rightarrow \infty \)?

Euler's Method

Euler's Method is a numerical process that approximates solutions to differential equations with initial values. Starting from the initial value \((t_0, y_0)\), one approximates the function by drawing a segment of the tangent line to the function (slope is determined by the differential equation, and the point is given) for a small distance \( \Delta t = h \). At the end of this segment, a new point can be used to determine a new tangent line using the point to determine a new slope. One can continue this process indefinitely; note that smaller steps \( h \) yield better approximations.
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example: Use Euler's Method to draw an approximate solution to the initial-value problem \( \frac{dy}{dt} = 2t - 3y + 1 \) that passes through the origin, using steps of \( \Delta t = 0.5 \).

Euler's Method: Approximate values for the solution of the initial-value problem
\[ y' = F(t, y), \quad y(t_0) = y_0, \] with step size \( h \), at \( t_{n+1} = t_n + h \), are

\[ y_{n+1} = y_n + h F(t_n, y_n) \quad n = 0, 1, 2, 3 \]

example: Look at the tables for example 3 in text.

example (7.3.24): Use Euler's Method with step size 0.2 to estimate \( y(1) \), where \( y(x) \) is the solution of the initial-value problem \( y' = xy - x^2, \quad y(0) = 1 \)

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