Chapter 7 - Differential Equations

7.1 Modeling with Differential Equations

A differential equation is an equation containing an unknown function and one or more of the derivatives of the function. The equation may contain higher-order derivatives. This chapter discusses finding a solution that satisfies the equation. One of the most common types of differential equations are those that model population growth.

The text looks at a growth model for population growth of yeast, which are single-celled organisms. A simple model assumes that each cell produces offspring at a constant rate \( \beta \), so the total rate of offspring production is \( \beta \cdot N(t) \), where \( N(t) \) is the number of cells at time \( t \). The loss rate of yeast cells is given by \( \mu \cdot N(t) \), where \( \mu \) is the constant death rate per cell. The rate of change is the production rate minus the death rate: \( \beta \cdot N(t) - \mu \cdot N(t) \). This leads to the differential equation

\[
\frac{dN(t)}{dt} = \beta \cdot N(t) - \mu \cdot N(t) = r \cdot N(t)
\]

where \( r = \beta - \mu \) is a constant, called the per capita growth rate. The differential equation has \( t \) as its independent variable and \( N \) as the dependent variable. Note that if \( r > 0 \), the growth rate \( \frac{dN}{dt} \) is positive and the population increases; if \( r < 0 \), the growth rate \( \frac{dN}{dt} \) is negative and the population decreases.

We will be solving differential equations in this chapter; which requires we find an explicit function that satisfies the differential equation. Note that the solution to the differential equation above, \( \frac{dN}{dt} = r \cdot N \) is \( N(t) = C \cdot e^{rt} \). One can find \( C \) by setting \( t = 0 \), so \( C = N(0) \) is the initial population. This type of model for population growth assumes that population grows indefinitely at a constant rate.

The data in the text example shows that yeast cells do not grow at a constant rate indefinitely. The per capita growth rate is not a constant but rather is (approximately) \( 0.55 - 0.0026N \). So a better model for the data in the example is the differential equation

\[
\frac{dN}{dt} = (0.55 - 0.0026N) \cdot N
\]
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This is an example of a logistic differential equation, which assumes that growth rate slows as the population approaches maximum population \( K \), called the carrying capacity. The general model for the differential equation of the logistic equation is

\[
\frac{dN}{dt} = r \left(1 - \frac{N}{K}\right)N,
\]

where \( r \) is the initial growth rate.

ex: Find the values of \( r \) and \( K \) for the model of yeast production

\[
\frac{dN}{dt} = (0.55 - 0.0026N)N.
\]

Note: there are two constant solutions to the general logistic equation, namely \( N = 0 \) and \( N = K \). Constant solutions to differential equations are called equilibrium solutions.

If \( 0 < N < K \), then \( \frac{dN}{dt} > 0 \) and the population increases; if \( N > K \), then \( \frac{dN}{dt} < 0 \) and the population decreases. In either case the population approaches the carrying capacity: \( N(t) \rightarrow K \) as \( t \rightarrow \infty \).

Notation: The order of a differential equation is the order of the highest derivative in the equation. A solution is a function that when substituted into the equation produces a equality. Usually a differential equation has many solutions; a particular solution can be found if given an initial condition. Finding a solution to a differential equation with a given initial condition is called an initial-value problem.

Pure-Time Differential Equations: \( \frac{dy}{dt} = f(t) \). These are the types of differential equations that we have been studying all quarter.

example: Solve the differential equation \( \frac{dy}{dx} = e^{2x} + 6\sqrt{x} \), where \( y = 5 \) when \( x = 0 \).

Autonomous Differential Equations: \( \frac{dy}{dt} = g(y) \). The (uninhibited) growth and the logistic differential equations are examples of autonomous differential equations.

Nonautonomous Differential Equations are a combination of pure-time and autonomous differential equations:

example: Verify that \( y(t) = Ce^{-t} + \frac{1}{2}(2 - \cos t + \sin t) \) is a solution to the (non-autonomous differential equation \( \frac{dy}{dt} = 1 + \sin t - y \).