7.1 Modeling with Diff. Eq. (Wrap-Up)

Logistic D.E.: \[ \frac{dN}{dt} = r \left(1 - \frac{N}{K}\right)N \]

\( N = N(t) \)

If \( N = \text{constant} \) then \( \frac{dN}{dt} = 0 \)

This solution is called an equilibrium solution.

\[ \frac{dN}{dt} = 0 = r \left(1 - \frac{N}{K}\right)N \]

\[ \frac{dN}{dt} > 0 \quad 1 - \frac{N}{K} = 0 \quad N = 0 \]

So logistic equation was two equilibria.

\( N = 0 \) \( N = K \) (carrying capacity).

\( \frac{dN}{dt} > 0 \) \( K \frac{dN}{dt} < 0 \)
IF \( N < K \) \( \frac{dN}{dt} > 0 \) \( \text{N INCREASING} \)

IF \( N > K \) \( \frac{dN}{dt} < 0 \) \( \text{N DECREASING} \)

\[ 1 \rightarrow 0 \rightarrow K \]

In long run: \( \lim_{t \to \infty} N(t) = K \).

**Notation!** Order of a D.E. is order of highest derivative in equation.

**Example:** \( \dot{t} \frac{dy}{dt} + 3 \int \frac{d^2y}{dt^2} y^2 = 0 \) D.E. of order 2

Solution for a D.E. is a function such that when substituted into D.E. equation is satisfied. A D.E. has many solutions, find a unique solution if given an initial condition.

Types of D.E. model: \( y = y(t) \)
Pure-time D.E., \( \frac{dy}{dt} = f(t) \) \( \text{WE HAVE SEEN THIS THROUGHOUT THE QUARTER.} \)

\(\begin{align*}
\frac{dy}{dt} &= t^2 - \sin(2t) + 3 \\
\end{align*}\)

\( y = \int \frac{dy}{dt} \, dt = \int (t^2 - \sin(2t) + 3) \, dt \)

\( y = \frac{1}{3} t^3 + \frac{\cos(2t)}{2} + 3t + C \bigg|_{t=0} = \frac{1}{2} + C = 5 \)

\( C = \frac{9}{2} \)

So \( y = \frac{1}{3} t^3 + \frac{1}{2} \cos(2t) + 3t + \frac{9}{2} \)

Autonomous D.E., \( \frac{dy}{dt} = g(y) \)

\(\begin{align*}
\frac{dN}{dt} &= r(1 - \frac{N}{K})N = g(N) \\
\end{align*}\)

Nonautonomous D.E., Combination of Pure Time \( / \) Autonomous, D.E.
Verify \( y(y(t)) = Ce^{-t} + \frac{1}{2}(2 - \cos t + \sin t) \)

is a solution to \( \frac{dy}{dt} = 1 + \sin t - y \)

\[
\frac{dy}{dt} = C[e^{-t}] + \frac{1}{2}(0 + \sin t + \cos t)
\]

\[
= -Ce^{-t} + \frac{1}{2}\sin t + \frac{1}{2}\cos t
\]

\[1 + \sin t - y = 1 + \sin t - (Ce^{-t} + \frac{1}{2}(2 - \cos t + \sin t))
\]

\[= Y + \sin t - Ce^{-t} / 1 + \frac{1}{2}\cos t - \frac{1}{2}\sin t
\]

\[= -Ce^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t
\]

7.2 Phase Plots, Equilibria, and Stability

Given an autonomous D.E. \( \frac{dy}{dt} = g(y) \)

(Note: Solution: \( y = y(t) \))

Phase Plot: Graph of \( g(y) \)
If \( \frac{dy}{dt} < 0 \) \( y \) is decreasing over time.

\[ \frac{dy}{dt} = 0 \]

**Logistic Equation**

\[ \frac{dN}{dt} = r \left( 1 - \frac{N}{K} \right) N \]

**Construct Phase Plot (Assume \( r > 0 \))**

\[ \frac{dN}{dt} + r \left( 1 - \frac{N}{K} \right) N = 0 \]

\( 1 - \frac{N}{K} = 0 \) when \( N = K \)

\( N < K \) \( 1 - \frac{N}{K} > 0 \)

\( N > K \) \( 1 - \frac{N}{K} < 0 \)

Given \( \frac{dy}{dt} = g(y) \) an equilibrium solution is a constant \( y \) such that \( \frac{dy}{dt} = 0 \) when \( y = y \)
An equilibrium $\bar{y}$ of a D.E. is **locally stable** if $y$ approaches $\bar{y}$ as $t \to \infty$ for all initial values of $y$ sufficiently close to $\bar{y}$.

$$\lim_{t \to \infty} y(t) = \bar{y} \iff \text{locally stable}$$

If $\bar{y}$ is not stable, then $\bar{y}$ is **unstable**.

**Example Logistic Equation:** \[ \frac{dN}{dt} = r \left(1 - \frac{N}{K}\right)N \]

Determine local stability of equilibria of this D.E.

- $N = 0$; $N = K$ equilibria.

**Graphical Illustration**

- $\frac{dN}{dt} > 0$ for $N < K$
- $\frac{dN}{dt} < 0$ for $N > K$
- $N = 0$, $N = K$ locally stable

$$\lim_{t \to \infty} N(t) = K \quad N = K \text{ locally stable}$$
Given D.E. of Allee effect:

\[ \frac{dN}{dt} = r(N-a)(1- \frac{N}{K})N \]

Draw phase plot; find equilibria, and determine the stability at each equilibrium.

N=K locally stable
N=0 " "
N=a locally unstable