4.3 \ L^1 \text{HOSPITAL'S RULE (WRAP-UP)}

If \( \frac{f(x)}{g(x)} \) is indeterminate of type \( \frac{0}{0} \)
(or \( \frac{\infty}{\infty} \)) as \( x \to a \) and if \( f \) and \( g \) are differentiable near \( x = a \) \( (g'(x) \neq 0 \) near \( a ) \) then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

Provided the limit exists.

For forms \( [\text{f(x)}]^n \to \infty , 0^0 , 1^0 \)

Key: Set \( y = f(x) \), find

\[
\ln(\ ) \left\downarrow \right. \ln(y) = \ln(f(x)) = g(x) \cdot \ln\left( \frac{f(x)}{g(x)} \right)
\]

Find \( \lim_{x \to a} \ln(y) = L \)

The \( \lim_{x \to a} y = e^L \)
\[ \lim_{{x \to 0^+}} (\tan x) \rightarrow 0 \]

\[ y = (\tan x)^x \]

Find \( \lim_{{x \to 0^+}} (y) \)

\[ \ln(y) = \ln((\tan x)^x) = x \cdot \ln(\tan x) \]

\[ \lim_{{x \to 0^+}} (\ln y) = \lim_{{x \to 0^+}} x \cdot \ln(\tan x) \]

\[ = \lim_{{x \to 0^+}} \frac{\ln(\tan x)}{\frac{1}{x}} \rightarrow -\infty \]

\[ \frac{\sec^2 x}{\frac{-1}{x^2}} \]

\[ \lim_{{x \to 0^+}} \frac{2x \cdot \sec^2 x}{1 + \tan x} \rightarrow 0 \]

\[ \lim_{{x \to 0^+}} \frac{[-2x] \cdot \sec^2 x + (-x^2)^2}{[2 \sec x \cdot \sec x \cdot \tan x]} = 0 \]

\[ \frac{\sec^2 x}{1} \]

\[ = 0 \]

So \( \lim_{{x \to 0}} \ln(y) = \lim_{{x \to 0}} x \cdot \ln(\tan x) = 0 \)
4.4. **Optimization Problems**

**Steps Listed in Text/Printed Notes**

**EX 300' of fencing. Enclose two rectangular corrals next to a river. (No fence required next to river), find dimensions that yield largest area of corrals.**

**Maximize Area (A)**

\[ A = l \cdot w \]

\[ A = (300 - 3w) \cdot w = f(w) \]

\[ A = 300w - 3w^2 \quad 0 \leq w \leq 100' \]

\[ \frac{dA}{dw} = 300 - 6w = 0 \]

50 = w

\[ \frac{d^2A}{dw^2} = -6 < 0 \quad \text{Max} \]

\[ l + 3w = 300 \]

\[ l = 300 - 3w \]

\[ w = 30' \]

\[ l = 210' \]

\[ w = 80' \]

\[ l = 60' \]
4.4 Optimization Problems

Steps in Solving Optimizations problems

1. Understand the problem. \((\text{READ; REREAD})\)
2. Draw a diagram.
3. Introduce notation for the quantity \(Q\) to be maximized or minimized, and for other values that can possibly vary in the problem.
4. Express \(Q\) in terms of some other symbols.
5. Find a formula for \(Q\) in terms of a single variable: \(Q = f(x)\).
6. Use calculus techniques to find the extreme value(s) of \(Q\).

example: A rancher with 300 feet of fencing material wants to enclose two rectangular corrals next to a river. The livestock don't swim, so no fence is needed along the river. Find the dimensions of the corrals that maximize the total area of the two corrals.

example: (4.4.6) The rate at which photosynthesis takes place for a species of phytoplankton is modeled by the function

\[
P = \frac{100I}{I^2 + I + 4}
\]

where \(I\) is the light intensity. For what light intensity is \(P\) a maximum?

example: (4.4.12) A box with a square base and open top must have a volume of 32,000 cm\(^3\). Find the dimensions of the box that minimizes the amount of material used.

example: Find the point(s) on the graph of \(y = x^2\) that is closest to the point \((0, 3)\).

example: (4.4.25) According to Fermat's Principle, a ray of light will travel from a point \(A\) in the air to a point \(B\) in the water by a path that minimizes the time taken. Show that for this to be true,

\[
\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}
\]
\[ A_{\text{max}} = (150')(50') = 7500 \text{ ft}^2 \]

(4.4.6) Given \( P = \frac{100I}{I^2 + I + 4} = f(I) \)

\( P = \text{photosynthesis} \)
\( I = \text{light intensity} \)

WHAT LIGHT INTENSITY YIELDS MAXIMUM \( P \)?

\[
\frac{dP}{dI} = \frac{[100](I^2 + I + 4) - 100I[2I + 1]}{(I^2 + I + 4)^2} = 0
\]

\[
\frac{100I^2 + 100I + 400 - 200I^2 - 100I}{(I^2 + I + 4)^2} = 0
\]

\[
\frac{dP}{dI} = \frac{-100I^2 + 400}{(I^2 + I + 4)^2} = 0
\]

\[
\text{Always +}
\]

\[
\frac{dP}{dI} > 0 \quad I < 2
\]

\[
\frac{dP}{dI} < 0 \quad I > 2
\]

\[
\frac{1}{2} = I
\]

\[ f(2) \text{ yields } A_{\text{max}} \]
Ex (4.4.12) A box with square base and open top must have volume of 32,000 cm³. Find dimensions of box that minimizes amount of material used.

**Minimize material used**

**Area** = \( A = \text{Base} + 4\text{sides} \)

\[
A = lw + 2lh + 2wh
\]

\[
A = x^2 + 2x(x)h + 2(x)h = x^2 + 4xh
\]

\[
V = l \cdot w \cdot h = (x)(x)h = 32000
\]

\[
h = \frac{32000}{x^2}
\]

\[
A = x^2 + \frac{4x}{1} \left( \frac{32000}{x^2} \right)
\]

\[
A = x^2 + \frac{128000}{x} = f(x)
\]

\( x > 0 \)
\[
\frac{dA}{dx} = 2x - \frac{128000}{x^2} = 0
\]

\[2x = \frac{128000}{x^2}\]

\[x = 64,000\]

\[x = \sqrt[3]{64,000} = 40\text{ cm.}\]

\[\frac{d^2A}{dx^2} = 2 + \frac{2 \cdot 128000}{x^3}\]

\[\left.\frac{d^2A}{dx^2}\right|_{x=40} = 2 + \frac{256000}{64000} > 0\]

So dimensions. \(l = 40\text{ cm}, h = 40\text{ cm}\)

\[h = \frac{32000}{x^2} = \frac{32000}{1600} = 20\text{ cm.}\]

\[\text{Min area is } (40^2 + 4(40)20\]

\[= 1600 + 3200 = 4800 \text{ cm}^2\]

Material.