3.8 (Wrap-up)

Linearization of \( f \) at \( x = a \):

\[
L(x) = f(a) + f'(a)(x-a)
\]

TAYLOR POLYNOMIALS

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\[
\begin{align*}
\text{Y} = \text{f}(x) \\
(a, \text{f}(a))
\end{align*}
\]
```

Linear (1st order) approx.

\[
L(x) = f(a) + f'(a)(x-a)
\]

"Best" parabola to approx. \( f(x) \) near \( x = a \)

Want to find a parabola \( P(x) = ax^2 + bx + c \)

Writes \( P(x) = A + B(x-a) + C(x-a)^2 \)

\[
\begin{align*}
\text{Want 1) } P(a) = f(a) &= A + B(0) + C(0)^2 = A \\
\text{2) } P'(a) = f'(a) &= B + 2C(a) = B
\end{align*}
\]

\[
P'(x) = 2x + 2a + 2C
\]
3) \( P'(a) = f'(a) \)

\[ P(x) = B + 2C(x-a) \]

\[ P''(x) = 0 + 2C[1] = 2C \]

\[ 2C = f''(a) \]

\[ C = \frac{f''(a)}{2} \]

So, "best fit" Parabola for \( f(x) \) Graph is:

\[ P(x) = f(a) + f'(a)(x-a) + \left( \frac{f''(a)}{2} \right)(x-a)^2 \]

For \( f(x) = \sqrt{x} \)

\[ f(x) = x^{\frac{1}{2}} = \sqrt{x} \]

\[ f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \]

\[ f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4(\sqrt{x})^3} \]

At \( x = 4 \),

\[ f(4) = \sqrt{4} = 2 \]

\[ f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \]

\[ f''(4) = -\frac{1}{4(\sqrt{4})^3} = -\frac{1}{32} \]

\[ P(x) = f(4) + f'(4)(x-4) + \left( \frac{f''(4)}{2} \right)(x-4)^2 \]
\[
\sqrt{x} \approx P(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2
\]

(Note: "Near" \( x = a \)

\[P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2\]

Is a good approx for \( f(x) \) [Better than \( f(a) \)]

Use \( P(x) \) to find an approx for \( \sqrt{5} \)

\[L(5) = 2 + \frac{1}{4}(5-4) = \frac{9}{4} = 2.25\]

\[P(5) = 2 + \frac{1}{4}(5-4) + \frac{1}{64}(5-4)^2 = 2 + \frac{1}{4} - \frac{1}{64} = \frac{128 + 16 \times \frac{1}{64}}{64} = \frac{143}{64} = \]

In general to approximate any fcns \( f(x) \) with a polynomial (of degree \( n \))

\[P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n\]

Want: \[P(a) = f(a)\] \[P'(a) = f'(a)\] \[P''(a) = f''(a)\]

Need \( f \) to be differentiable (\( n \) times)
\[ P^{(n)}(a) = f^{(n)}(a) \]

In this case,

\[ A_k = \frac{f^{(k)}(a)}{k!} \]

**Factorial**

\[ k! = 1 \cdot 2 \cdot 3 \cdots k \]

**DFN:** \( n \text{th Taylor Polynomial of } f \text{ near } x = a \)

\[ T_n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \]

\[ n! = 1 \cdot 2 \cdot 3 \cdot 4 \]

**Ex:** Find \( 4 \text{th Taylor Polynomial for} \)

\[ f(x) = \cos x \text{ at } x = 0 \]

\[ f(x) = \cos x \quad = 1 \]

\[ f'(x) = -\sin x \quad = 0 \]

\[ f''(x) = -\cos x \quad = -1 \]

\[ f'''(x) = \sin x \quad = 0 \]

\[ f^{(4)}(x) = \cos x \quad = 1 \]

\[ x = 0 \]

\[ T_4(x) = 1 + 0(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{0}{3!}(x-0)^3 + \frac{-1}{4!}(x-0)^4 \]

\[ = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \]
4.1 Maximum; Minimum Values

**Definition:** A function $f$ has an **absolute maximum** (global max) at $x = c$ if $f(c) \geq f(x)$ for all $x \in$ domain of $f$.

$f(c)$ is the **maximum value** of $f$.

- $f$ has an **absolute minimum** at $x = d$ if $f(d) \leq f(x)$ for all $x \in$ domain of $f$.

**Graphical Illustration:**

- $x = c$ is a point where $f$ achieves its **maximum** value $f_{\text{max}} = f(c)$.
- $x = d$ is a point where $f$ achieves its **minimum** value $f_{\text{min}} = f(d)$.

**Definition:** $f$ has a **local maximum** (relative max) at $x = e$ if $f(e) \geq f(x)$ in some region containing $e$.

[Also define a local minimum]
\[ f(x) = 8 \sin x \]
\[ f(x) = x^3 \]

**Example:**

- \( f_{\text{max}} = 1 \) at \( x = \pi / 2 \)
- \( f_{\text{min}} = -1 \) at \( x = -\pi / 2, 3\pi / 2, 5\pi / 2 \)

- \( f(x) = x^3 \)
  - No maximum
  - No minimum

**Example:**

- \( f(x) = x^3 \) on \([-2, 3]\)
  - \( f_{\text{max}} = 27 \) at \( x = 3 \)
  - \( f_{\text{min}} = -8 \) at \( x = -2 \)

**Theorem (Extreme Value Theorem):** If \( f \) is continuous on a closed interval \([a, b]\) then \( f \) has both an absolute maximum and an absolute minimum for some number \( c, d \in [a, b] \).
**Thm (Fermat)** If $f$ has a local extreme (max or min) at $x = c$, then $f'(c) = 0$ or $f'(c)$ D.N.E.

**Dfn** A critical number for $f$ is a value $x = c$ (in domain) where $f'(c) = 0$ or $f'(c)$ D.N.E.

**Ex** $f(x) = \sqrt{1 - x^2} = (1 - x^{-2})^{1/2}$, Domain $= \mathbb{R}$

$$f'(x) = \frac{1}{3}(1 - x^2) \left[-2x\right] = \frac{-2x}{3(1 - x^2)^{2/3}}.$$  
   @ $x = 0$ D.N.E., @ $x = \pm 1$

**Crit. No.** $x = 0, 1, -1$. 