Formulas/definitions for the final exam

\[ Ax + By + C = 0 \quad (y - y_0) = m(x - x_0) \quad y = m \cdot x + b \]

If \( ax^2 + bx + c = 0 \), then \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

\[ \log_b(M) = t \iff b^t = M \quad \log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N) \]

\[ \log_a(a^x) = x \quad \log_a(x^b) = bx \]

If \( \lim_{x \to a} \ln(y) = L \), then \( \lim_{x \to a} (y) = e^L \)

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad f'(c) = \frac{f(b) - f(a)}{b - a} \]

\[ \sin^2 \theta + \cos^2 \theta = 1 \quad c_k = \frac{f^{(k)}(a)}{k!} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

1. Find the (first) derivative \([ f'(x) \text{ or } \frac{dy}{dx} ]\): simplify your results.
   a. \( f(x) = 2x^3 \cdot \tan^{-1}(x^3) - \ln(1 + x^6) \)
   b. \( y = \ln(\sec(3x) + \tan(3x)) \)

2. Sketch the graph of an example of a function \( y = f(x) \) defined for all real \( x \) that satisfies all of the following conditions, and show any horizontal and/or vertical asymptotes with dotted lines:
   \[ f(0) = -3 \quad f(2) = 4 \quad \lim_{x \to 0^+} f(x) = 1 \quad \lim_{x \to 0^-} f(x) = -4 \]
   \[ \lim_{x \to -2^-} f(x) = -1 \quad \lim_{x \to -2^+} f(x) = -\infty \quad \lim_{x \to \infty} f(x) = 2 \quad \lim_{x \to (-\infty)} f(x) = \infty \]

3. Evaluate the following limit, if it exists (if the limit does not exist, write DNE). Justify your answers using algebraic techniques and limit laws.
   \[ \lim_{x \to 3} \left( \frac{1}{3x - 9} - \frac{2}{x^2 - 9} \right) \]
4. Given the equation \(x^4 + x^2y + y^2 = 48\):
   a. Find the point on the graph of this equation in the first quadrant whose \(x\)-coordinate is 2.
   b. Find the formula for \(\frac{dy}{dx}\) (in terms of both \(x\) and \(y\)).
   c. Find the equation, in slope-intercept form, of the line tangent to the graph of this equation at the point from part a.

5. Given the function \(f(x) = \sqrt[3]{x}(x + 8)\):
   a. Find the critical number(s) for the function.
   b. Find the interval(s) where \(f(x)\) is decreasing (if there is no interval, write NONE).
   c. Find any point(s) of inflection (If there is no point, write NONE).

6. Find the absolute maximum and minimum values of the function \(f(x) = x^3 - 18x^2 + 81x + 45\) on the interval \([-1, 10]\) state the value(s) of \(x\) where each of these extremes occur. Justify your answers using calculus.

7. Assume that \(x\) and \(y\) are differentiable functions of \(t\). Use implicit differentiation to find \(\frac{dy}{dt}\) when \(x^2 - y^3 = 5x + 2\), \(\frac{dx}{dt} = 4\) for \(x = 3\) (note that you will need to find \(y\)).

8. A sample of a radioactive material decays to 75% of its original amount after one year.
   a. What is the half life of this material? Write either the exact value, or write an approximation correct to the nearest tenth of a year.
   b. How long will it take a sample to decay to 20% of its original amount? Write either the exact value, or write an approximation correct to the nearest tenth of a year.

9. Use the formula \(f(x) \approx L(x) = f(a) + f'(a)(x - a)\) to find a fraction (of integers) that approximates \(\sqrt{78}\). Be sure to identify the function \(f(x)\) and the value of \(a\) that is used.

10. Use Newton's method to find \(x_2\) and \(x_3\), the approximations to the solution of the equation \(x^3 - 9x + 9 = 0\) that lies between \(x = 2\) and \(x = 3\). Start with \(x_1 = 3\). Express your answers as fractions of integers or decimals correct to three decimal places.

11. Evaluate the following limits, if they exist (if a limit does not exist, write DNE). Justify your answers using algebraic techniques and limit laws.
   a. \(\lim_{x \to 1} \frac{\cos(\pi x) + 1}{x - 1 - \ln(x)}\)
   b. \(\lim_{x \to \infty} x^2 \cdot \sin\left(\frac{\pi}{x^2}\right)\)

12. A right triangle has its base on the \(x\)-axis, one vertex at the origin, another on the positive \(x\)-axis, and its upper vertex on the parabola \(y = 18 - 2x^2\). What are the dimensions of the triangle with the largest area? What is the largest area?
1. a. $6x^2\tan^{-1}(x^3)$  
   b. $3\sec(3x)$

2. graph

3. $\frac{1}{18}$

4. a. $(2, 4)$  
   b. $\frac{dy}{dx} = -\frac{4x^3 + 2xy}{x^2 + 2y}$  
   c. $y = -4x + 12$

5. a. $0, -2$  
   b. $(-\infty, -2)$  
   c. $(0,0); \left(4, 12\sqrt[3]{4}\right)$

6. $f_{\text{max}} = 153 \ @ \ x = 3$  
   $f_{\text{min}} = -55 \ @ \ x = -1$

7. $\frac{dy}{dt} = \frac{1}{3}$

8. a. $\frac{\ln(0.5)}{\ln(0.75)} \approx 2.4 \text{ years}$  
   b. $\frac{\ln(0.2)}{\ln(0.75)} \approx 5.6 \text{ years}$

9. $f(x) = \sqrt{x} \quad a = 81 \quad \sqrt{78} \approx \frac{107}{36}$

10. $x_2 = \frac{5}{2} = 2.5$  
    $x_3 = \frac{178}{78} \approx 2.282$

11. a. $\pi^2$  
    b. $\pi$

12. base $= \sqrt{3}$  
    height $= 12$  
    area $= 6\sqrt{3}$