4.5 Recursions: Equilibria and Stability

Recall: (from sec. 1.6) recursive sequences -- \( a_{n+1} = f(a_n) \); we call \( f \) the updating function. We discussed long-term behavior of these sequences in section 2.1.

Definition: An **equilibrium** of a recursive sequence \( x_{t+1} = f(x_t) \) is a number \( \hat{u} \) that is left unchanged by the function \( f \); that is \( f(\hat{u}) = \hat{u} \). An equilibrium is also called a **fixed point** of \( f \) because \( f \) leaves the point \( \hat{u} \) fixed.

example: if \( f(x) = \frac{x+8}{3} \), then \( x = 4 \) is an equilibrium.

To find equilibria algebraically, solve the equation \( f(x) = x \) for \( x \). To locate equilibria geometrically, find where the graphs of \( y = f(x) \) and \( y = x \) intersect.

Definition: An equilibrium is called **stable** if solutions that begin close to the equilibrium approach that equilibrium (so if \( \hat{u} \) is an equilibrium for the recursive equation \( x_{t+1} = f(x_t) \) and \( x_0 \) is sufficiently close to \( \hat{u} \), then \( \lim_{t \to \infty} x_t \to \hat{u} \) as \( x \to \infty \)). It is **unstable** if solutions that start close to the equilibrium move away from it.

**Cobwebbing**: a graphical method for finding equilibria. See text pp. 300-304.

**Stability Criterion**: Suppose that \( \hat{u} \) is an equilibrium of the recursive sequence \( x_{t+1} = f(x_t) \), where \( f' \) is continuous. If \( |f'(\hat{u})| < 1 \), the equilibrium is stable; if \( |f'(\hat{u})| > 1 \), the equilibrium is unstable.

example (4.5.ex4): Use the stability criterion to discuss the stability of the logistic difference equation \( x_{t+1} = c x_t (1 - x_t) \).

example (4.5.8): Find the equilibria and classify as stable or unstable for the difference equation \( x_{t+1} = \frac{3x_t}{1+x_t} \).

example (4.5.22): Find the equilibria and classify as stable or unstable for the difference equation \( x_{t+1} = \frac{375}{x_t - 90} + 100 \).