3.8 Linear Approximations and Taylor Polynomials

Definition: The linearization of \( f \) at \( a \) (the tangent line approximation) is the linear function whose graph is the tangent line to \( f \) at the point \((a, f(a))\), and is given by the formula

\[
L(x) = f(a) + f'(a)(x - a)
\]

example: If \( f(x) = \sqrt[3]{x} \), find the linearization of \( f \) at \( a = 8 \).

example: (3.R.91 part a) If duration of time, \( t \), required to remove urea from the blood is given by the equation \( t = t(c) = \ln\left(\frac{3c + \sqrt{9c^2 - 8c}}{2}\right) \), find the linear approximation of \( t \) near \( c = 1 \).

example: Use the linearization of \( f(x) = \sqrt{x} \) to find an approximate value of: \( \sqrt{5} \); \( \sqrt{3.7} \); \( \sqrt{4.02} \)


Note that slope of tangent line at \( x_1 \) is \( m_{tan} = f'(x_1) \). It crosses the \( x \)-axis at \((x_2, 0)\), so slope of the tangent line is also \( m_{tan} = \frac{-f(x_1)}{x_2 - x_1} \). Solving for \( x_2 \) yields: \( x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \). Repeat this procedure with the point \((x_2, f(x_2))\).

In general, \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \). If all goes well, we have \( \lim_{n \to \infty} x_n = r \).

example: Find an approximation for \( \sqrt{5} \). Let \( f(x) = x^2 - 5 \). Use

\[
x_{n+1} = x_n - \frac{(x_n^2 - 5)}{2x_n}
\]
3.8 Linear Approximations and Taylor Polynomials (continued)

example (text example 3.8.6): Find the solution to the equation \( \cos x = x \)

Taylor Polynomials

The tangent line approximation \( L(x) \) is the best linear (first-degree) approximation to \( f(x) \) at \( x = a \) since the functions \( f \) and \( L \) have the same function value and the same slope at \( x = a \). For a better approximation, we will look for a quadratic (second-degree) function \( P(x) \) to approximate \( f(x) \). We will want:

\[
P(a) = f(a) \quad P'(a) = f'(a) \quad P''(a) = f''(a)
\]

Procedure: Let \( P(x) = A + B(x - a) + C(x - a)^2 \). Find \( A, B, \) and \( C \) using the equations above.

example: Find the second degree Taylor polynomial for \( f(x) = \sqrt{x} \) at \( x = 4 \).

For better approximations, continue this process. To approximate \( f(x) \) with an \( n^{th} \) degree polynomial: \( T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \ldots + c_n(x - a^n) \), set the first \( n \) derivatives of \( T_n \) equal to the same derivatives of \( f(x) \). This process yields that \( c_k = \frac{f^{(k)}(a)}{k!} \), where \( k! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k \), \( (k \text{ factorial}) \)

Definition: The \( n^{th} \) Taylor polynomial of \( f \) at \( x = a \) is the polynomial

\[
T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a^n)
\]

example: Find the \( 4^{th} \) Taylor polynomial, \( T_4(x) \), for \( f(x) = \cos x \) at \( x = 0 \).