3.6 Exponential Growth and Decay

Note that the change in a quantity often depends on the quantity itself.

Such a quantity $y$ satisfies: \[ \frac{dy}{dt} = ky \] (growth if $k > 0$, decay if $k < 0$)

Theorem: The only solutions of the differential equation \[ \frac{dy}{dt} = ky \] are the exponential functions $y = y(t) = y(0)e^{kt}$.

Population Growth: $P = \text{population at time } t$; then \[ \frac{dP}{dt} = kP, \] where $k = \frac{1}{P} \frac{dP}{dt}$ \textbf{relative growth rate} (growth rate is $k \cdot 100\%$)

example (Ex 1 in text) World pop'n: 2560 million in 1950, 3040 million in 1960

\[ P(t) = P(0)e^{kt} = 2560e^{kt} \] $t =$ years since 1950 Solve for $k$

\[ k = \frac{1}{10} \ln \left( \frac{3040}{2560} \right) \approx .017185 \] [World pop'n on 26 Oct 2016 $\approx 7460$ million]

Radioactive Decay: $m = m(t) = \text{radioactive mass remaining at time } t$; then

\[ \frac{dm}{dt} = km \] (here $k < 0$), so $m(t) = m_0e^{kt}$

Note that the \textbf{half-life} is the time required for half of the quantity to decay.

example: (3.6.7) The half-life of cesium-137 is 30 years. Suppose we have a 100 $mg$ sample.

a. Find the mass that remains after $t$ years.

b. How much of the sample remains after 100 years?

c. After how long will only 1 $mg$ of the sample remain?

\textbf{Newton's Law of Cooling}: $T(t) =$ temperature of an object at time $t$. Rate at which the object cools (or heats up) is proportional to the difference of the object and its surrounding temperature: \[ \frac{dT}{dt} = k(T - T_s). \] Solving this formula yields: $T = T_s + T_d e^{kt}$, where $T_d$ is the initial temperature difference.