3.5 The Chain Rule

example: $f(x) = e^{2x} = e^x \cdot e^x$  

example: $y = (x^2 + 7x)^3$

The Chain Rule: If $f$ and $g$ are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F$ is differentiable and $F'$ is given by the product $F'(x) = f'(g(x)) \cdot g'(x)$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

example: $y = (x^3 - 4x^2 + 11)^5$  

example: $F(x) = \tan\left(5x^2 - \frac{\pi}{6}\right)$

example: $G(x) = \left(\frac{2x-3}{5x+4}\right)^7$  

examples 3.5.47 ; 3.5.51

The (generalized) Power Rule: If $n$ is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = n u^{n-1} \frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}(g(x)^n) = n (g(x))^{n-1} \cdot g'(x)$$

example: $H(x) = \frac{1}{\sqrt[3]{x^2-5x+7}}$  

example: $y = e^{\sin x}$

Note: since $a = e^{\ln a}$, we have $a^x = (e^{\ln a})^x = e^{\ln a \cdot x}$

Hence, $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{\ln a \cdot x}) = (e^{\ln a \cdot x})[\ln a \cdot x]' = (a^x) \ln a$

example: $f(x) = 4^x$ ; find $f'(x)$

Recall from section 3.3, $\frac{d}{dx}(2^x) \approx 2^x (.69)$  

$\frac{d}{dx}(3^x) \approx 3^x (1.098)$

$\frac{d}{dx}(2^x) = 2^x (\ln 2)$  

$\frac{d}{dx}(3^x) = 3^x (\ln 3)$
3.5 The Chain Rule (continued)

Implicit Differentiation

If \( y = f(x) \) [i.e. \( y \) is defined \textit{explicitly} in terms of \( x \)], we have formulas.

Consider an equation with expressions in \( x \) and \( y \). Then \( y \) is defined \textit{implicitly} in terms of \( x \). Since \( y \) varies when \( x \) does, what is \( \frac{dy}{dx} \)?

example: \( x^2 + y^2 = 36 \) \( \rightarrow \) \( y = \pm \sqrt{36 - x^2} \)

\[ x^2 + (f(x))^2 = 36 \]

Differentiate both sides of the equation with respect to \( x \):

\[ \frac{d}{dx}(x^2 + (f(x))^2) = \frac{d}{dx}(36) \]

\[ 2x + 2(f(x))f'(x) = 0 ; \quad \text{Solve for } f'(x) = \]

Method: differentiate both sides w.r.t. \( x \); when taking the derivative of a quantity involving \( y \) use the chain rule. Then solve for \( \frac{dy}{dx} \).

examples: \( x^3 + y^4 = x^2y^2 \)

\( y^5 + x^2y^3 = 1 + ye^{x^2} \)

example: \( \tan(2x - 3y) = \frac{y}{1+x^2} \)

example: Find the equation of the tangent line to the curve defined by the equation \( x^2 + 2xy - y^2 + x = 2 \) at the point \( (1, 2) \).

Related Rates: Use implicit differentiation show how rate are realated.

example: If \( V = \frac{4}{3}\pi r^3 \) and \( \frac{dV}{dt} = 7 \text{ cm}^3/\text{sec} \), find \( \frac{dr}{dt} \) when \( r = 6 \text{ cm} \).