2.2 Limits of Functions at Infinity

Definition: Suppose $f$ is defined on $(a, \infty)$; then \( \lim_{x \to \infty} f(x) = L \) means that the values of $f(x)$ can be made arbitrarily "close" to $L$ if $x$ is sufficiently "large". [Note the definition for \( \lim_{x \to -\infty} f(x) = L \) is similar]

Definition: The line $y = L$ is a \textbf{horizontal asymptote} of $f(x)$ if either

\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.
\]

Examples: 1) $f(x) = \frac{2x-5}{x+1}$ 2) $f(x) = \frac{1}{x}$

Note: if $p > 0$, then \( \lim_{x \to \infty} \left( \frac{1}{x^p} \right) = 0 \)

[also \( \lim_{x \to -\infty} \left( \frac{1}{x^p} \right) = 0 \) when the fraction is defined]

Examples: 1) rational functions 2) \( \lim_{x \to \infty} \left( \sqrt{x^2 + 4x + 100} - x \right) \)

**Limits involving Exponential Functions**: By examining the graphs of $f(x) = b^x$, one can conclude that

1) if $0 < b < 1$, then \( \lim_{x \to \infty} b^x = 0 \) 2) if $b > 1$, then \( \lim_{x \to -\infty} b^x = 0 \)

In the special case where $b = e$, we have the following limits:

1) \( \lim_{x \to \infty} e^{-x} = 0 \) 2) \( \lim_{x \to -\infty} e^x = 0 \)

Examples: a) \( \lim_{x \to -\infty} (1 - e^x) \) b) (ex. 2.2.8) \( \lim_{t \to \infty} \left( \frac{64}{1 + 31e^{-0.7944t}} \right) \)

Note: \( \lim_{x \to \infty} f(x) = \infty \) means $f(x)$ becomes large when $x$ becomes large.

Example: $f(x) = x^3$

Beware!! -- cannot do arithmetic with infinite values. example: \( \lim_{x \to \infty} \left( \frac{x^2}{x} \right) \)