HW DUE FRIDAY: § 1.2, 1.4, 1.5

**2.1 LIMITS OF SEQUENCES (CONT'D)**

**DEF:** A sequence \( \{a_n\} \) has a limit \( L \), denoted

\[ \lim_{n \to \infty} a_n = L \]

if one can make terms \( a_n \) as close to \( L \) as one likes, by taking \( n \) to be sufficiently large. If a limit exists, say the sequence converges to \( L \), otherwise the sequence diverges.

**Ex.**

(2) \( a_n = \frac{n}{2^{n-1}} \)

Last time \( \lim_{n \to \infty} \frac{n}{2^{n-1}} = \frac{1}{2} \)

Want show \( \frac{n}{2^{n-1}} \) is "close" to \( \frac{1}{2} \) for large \( n \).

So want

\[ \left| \frac{n}{2^{n-1}} - \frac{1}{2} \right| < \frac{1}{100} \]

\[ \left| \frac{2n - (2^{n-1})}{2(2^{n-1})} \right| = \left| \frac{1}{2^{n-1}} \right| \]
\[
\begin{align*}
\text{Want} & \quad \frac{1}{2(2n-1)} < \frac{1}{100} \cdot \frac{100}{2(2n-1)} \\
& \quad 100 < 4n - 2 \\
& \quad 102 < 4n \\
& \quad 25.5 = \frac{102}{4} < n \\
\text{For } n \geq 26 & \quad \left| \frac{n}{2n-1} - \frac{1}{2} \right| < \frac{1}{100} \\
\text{ex. b) } b_n &= (-1)^n \\
& \quad b_1 = -1^{(1)} = -1 \\
& \quad b_2 = (-1)^{(2)} = 1 \\
& \quad b_3 = (-1)^{(3)} = -1 \\
\text{-1, 1, -1, 1, -1, 1, ...} & \quad \text{Does } b_n \text{ get close to ANY ONE VALUE?} \\
\lim_{n \to \infty} (-1)^n & \quad \text{Does NOT EXIST} \\
\text{So } \{ (-1)^n \} & \quad \text{DIVERGES} \\
\text{ex. } f_n & \quad \{ 1, 1, 2, 3, 5, 8, 13, 21, ... \} \\
& \quad \lim_{n \to \infty} f_n = \infty \\
\text{So } \{ f_n \} & \quad \text{DIVERGES.}
\end{align*}
\]
LIMIT LAWS \{a_n\}, \{b_n\} CONVERGENT SEQUENCES.

\[ \text{CEIR \ CONSTANT, \ THEN:} \]

1. \[ \lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = L + M \]

2. \[ \lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n = L - M \]

3. \[ \lim_{n \to \infty} c \cdot a_n = c \cdot \lim_{n \to \infty} a_n = c \cdot L \]

4. \[ \lim_{n \to \infty} (a_n \cdot b_n) = L \cdot M \]

5. \[ \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \frac{L}{M} \quad \text{AS \ LONG, \ AS} \ b_n \neq 0 \]
\[ \text{AND} \ M \neq 0 \]

6. \[ \lim_{n \to \infty} (a_n)^p = L^p \quad \text{ex} \quad \lim_{n \to \infty} \left( \frac{n}{2n-1} \right)^3 = \left( \frac{1}{2} \right)^3 = \frac{1}{8} \]

7. \[ \lim_{n \to \infty} \frac{1}{n^p} = 0 \quad \text{IF} \ p > 0 \]

\[ \left\{ \frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots \right\} = \left\{ \frac{1}{n^2} \right\} \rightarrow 0 \]
**DEF GEOMETRIC SEQUENCE IS OF FORM**

\[ b_n = a \cdot r^n, \text{ starting with } b_0 = a. \]

\[ a, ar, ar^2, ar^3, ar^4, \ldots \]

**EX**

\[ 3, 6, 12, 24, 48, 96, \ldots \]

\[ b_n = 3 \cdot 2^n \]

**EX**

\[ 12, 6, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots \]

\[ b_n = 12 \cdot \left(\frac{1}{2}\right)^n \]

**NOTE**

\[
\lim_{n \to \infty} r^n = \begin{cases} 
0 & 0 < r < 1 \\
1 & r = 1 \\
\infty & r > 1 
\end{cases}
\]

**EX**

\[
\lim_{n \to \infty} \left(\frac{3n^2 - 5n + 8}{5n^2 + 7n + 9}\right) \div \frac{n^2}{n^2} = \lim_{n \to \infty} \frac{3n^2}{5n^2} - 5x + 8 = \infty
\]

\[
\lim_{n \to \infty} \frac{3n^2}{5n^2 + 7n + 9} = -5 \lim_{n \to \infty} \frac{1}{n} = -5 \cdot 0 = 0
\]

\[
\lim_{n \to \infty} 3 - \frac{5}{n} + \frac{8}{n^2} = \lim_{n \to \infty} \left(3 - \frac{1}{n} + \frac{8}{n^2}\right)
\]

\[
= \frac{3+0+0}{5+0+0} = \frac{3}{5}
\]
\[
\begin{align*}
\lim_{{n \to \infty}} \frac{4^n + 1}{8^n} &= \lim_{{n \to \infty}} \left( \frac{4^n}{8^n} + \frac{1}{8^n} \right) \\
&= \lim_{{n \to \infty}} \left[ \left( \frac{4}{8} \right)^n + \left( \frac{1}{8} \right)^n \right] \\
&= \lim_{{n \to \infty}} \left( \frac{1}{2} \right)^n + \lim_{{n \to \infty}} \left( \frac{1}{8} \right)^n = 0
\end{align*}
\]

\[
\begin{align*}
\lim_{{n \to \infty}} \frac{7^n + 9^n}{10^n} &= \lim_{{n \to \infty}} \left( \frac{7}{10} \right)^n + \lim_{{n \to \infty}} \left( \frac{9}{10} \right)^n \\
&= 0 + 0 = 0
\end{align*}
\]

**Example (Example 2.1.5 in Text)** Given: \( \frac{C_{n+1}}{C_n} = 0.3C_n + 0.2 \)

With \( C_0 = 0 \) Find:

\( a) \) \( C_4 \)

\( b) \) \( \lim_{{n \to \infty}} C_n \) (if possible)

\( C_1 = 0.3(0) + 0.2 = 0.2 \)

\( C_2 = 0.3(0.2) + 0.2 = 0.06 + 0.2 = 0.26 \)

\( C_3 = 0.3(0.26) + 0.2 = 0.078 + 0.2 = 0.278 \)

\( C_4 = 0.3(0.278) + 0.2 = \boxed{0.278} \)
b) \( \lim_{n \to \infty} C_n \quad \text{assumes this exists,} \quad \lim_{n \to \infty} C_n = L \)

**Use**

\[ C_{n+1} = 0.3C_n + 0.2 \]

**Take limit of both sides**

\[
\lim_{n \to \infty} C_{n+1} = \lim_{n \to \infty} (0.3C_n + 0.2)
\]

\[
\lim_{n \to \infty} C_{n+1} = 0.3 \lim_{n \to \infty} C_n + \lim_{n \to \infty} (0.2)
\]

\[ 1 \cdot L = 0.3L + 0.2 \]

\[ -0.3L = -0.3L \]

\[ 0.7L = 0.2 \rightarrow L = \frac{2}{0.7} = \frac{2}{7} \]

**Geometric Series; Successive Sums w/ Terms Are from a Geo. Sequence**

\[ S_0 = a \]
\[ S_1 = a + ar \]
\[ S_2 = a + ar + ar^2 \]
\[ \vdots \]
\[ S_n = a + ar + ar^2 + \ldots + ar^n \]
\[
S_n = \frac{a(1 - r^n)}{1 - r}
\]

\[
S_n - rS_n = a - ar^{n+1}
\]

\[
(1 - r) S_n = a(1 - r^{n+1})
\]

**Sum of Geometric Series**

\[
S = a + ar + ar^2 + \ldots
\]

\[
\lim_{n \to \infty} \sum_{i=0}^{n} ar^i = \lim_{n \to \infty} a \left( \frac{1 - r^{n+1}}{1 - r} \right) \quad \text{If } |r| < 1
\]

\[
S_n = \frac{a}{1 - r}
\]