Chapter 22
Value at Risk
The Question Being Asked in VaR

“What loss level is such that we are $X\%$ confident it will not be exceeded in $N$ business days?”
**VaR vs. Expected Shortfall**

- VaR is the loss level that will not be exceeded with a specified probability.
- Expected Shortfall (or C-VaR) is the expected loss given that the loss is greater than the VaR level.
- Although expected shortfall is theoretically more appealing, it is VaR that is used by regulators in setting bank capital requirements.
Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”
Historical Simulation to Calculate the One-Day VaR

- Create a database of the daily movements in all market variables.
- The first simulation trial assumes that the percentage changes in all market variables are as on the first day.
- The second simulation trial assumes that the percentage changes in all market variables are as on the second day.
- and so on.
Historical Simulation continued

- Suppose we use 501 days of historical data (Day 0 to Day 500)
- Let $v_i$ be the value of a variable on day $i$
- There are 500 simulation trials
- The $i$th trial assumes that the value of the market variable tomorrow is

$$v_{500} \frac{v_i}{v_{i-1}}$$
**Example: Calculation of 1-day, 99% VaR for a Portfolio on Sept 25, 2008 (Table 22.1, page 498)**

<table>
<thead>
<tr>
<th>Index</th>
<th>Value ($000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DJIA</td>
<td>4,000</td>
</tr>
<tr>
<td>FTSE 100</td>
<td>3,000</td>
</tr>
<tr>
<td>CAC 40</td>
<td>1,000</td>
</tr>
<tr>
<td>Nikkei 225</td>
<td>2,000</td>
</tr>
</tbody>
</table>
## Data After Adjusting for Exchange Rates

*(Table 22.2, page 498)*

<table>
<thead>
<tr>
<th>Day</th>
<th>Date</th>
<th>DJIA</th>
<th>FTSE 100</th>
<th>CAC 40</th>
<th>Nikkei 225</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Aug 7, 2006</td>
<td>11,219.38</td>
<td>11,131.84</td>
<td>6,373.89</td>
<td>131.77</td>
</tr>
<tr>
<td>1</td>
<td>Aug 8, 2006</td>
<td>11,173.59</td>
<td>11,096.28</td>
<td>6,378.16</td>
<td>134.38</td>
</tr>
<tr>
<td>2</td>
<td>Aug 9, 2006</td>
<td>11,076.18</td>
<td>11,185.35</td>
<td>6,474.04</td>
<td>135.94</td>
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<tr>
<td>3</td>
<td>Aug 10, 2006</td>
<td>11,124.37</td>
<td>11,016.71</td>
<td>6,357.49</td>
<td>135.44</td>
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<tr>
<td>...</td>
<td>.....</td>
<td>.....</td>
<td>.....</td>
<td>.....</td>
<td>.....</td>
</tr>
<tr>
<td>499</td>
<td>Sep 24, 2008</td>
<td>10,825.17</td>
<td>9,438.58</td>
<td>6,033.93</td>
<td>114.26</td>
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<tr>
<td>500</td>
<td>Sep 25, 2008</td>
<td>11,022.06</td>
<td>9,599.90</td>
<td>6,200.40</td>
<td>112.82</td>
</tr>
</tbody>
</table>
**Scenarios Generated**  
*(Table 22.3, page 499)*

\[
11,022.06 \times \frac{11,173.59}{11,219.38} = 10,977.08
\]

<table>
<thead>
<tr>
<th>Scenario</th>
<th>DJIA</th>
<th>FTSE 100</th>
<th>CAC 40</th>
<th>Nikkei 225</th>
<th>Portfolio Value ($000s)</th>
<th>Loss ($000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10,977.08</td>
<td>9,569.23</td>
<td>6,204.55</td>
<td>115.05</td>
<td>10,014.334</td>
<td>−14.334</td>
</tr>
<tr>
<td>2</td>
<td>10,925.97</td>
<td>9,676.96</td>
<td>6,293.60</td>
<td>114.13</td>
<td>10,027.481</td>
<td>−27.481</td>
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<tr>
<td>3</td>
<td>11,070.01</td>
<td>9,455.16</td>
<td>6,088.77</td>
<td>112.40</td>
<td>9,946.736</td>
<td>53.264</td>
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<td>........</td>
<td>........</td>
<td>........</td>
<td>........</td>
<td>........</td>
<td>........</td>
</tr>
<tr>
<td>499</td>
<td>10,831.43</td>
<td>9,383.49</td>
<td>6,051.94</td>
<td>113.85</td>
<td>9,857.465</td>
<td>142.535</td>
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<tr>
<td>500</td>
<td>11,222.53</td>
<td>9,763.97</td>
<td>6,371.45</td>
<td>111.40</td>
<td>10,126.439</td>
<td>−126.439</td>
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</tbody>
</table>
## Ranked Losses  
*Table 22.4, page 500*

<table>
<thead>
<tr>
<th>Scenario Number</th>
<th>Loss ($000s)</th>
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</thead>
<tbody>
<tr>
<td>494</td>
<td>477.841</td>
</tr>
<tr>
<td>339</td>
<td>345.435</td>
</tr>
<tr>
<td>349</td>
<td>282.204</td>
</tr>
<tr>
<td>329</td>
<td>277.041</td>
</tr>
<tr>
<td>487</td>
<td>253.385</td>
</tr>
<tr>
<td>227</td>
<td>217.974</td>
</tr>
<tr>
<td>131</td>
<td>205.256</td>
</tr>
</tbody>
</table>

99% one-day VaR
The \( N \)-day VaR

- The \( N \)-day VaR for market risk is usually assumed to be \( \sqrt{N} \) times the one-day VaR.
- In our example the 10-day VaR would be calculated as
  \[
  \sqrt{10} \times 253,385 = 801,274
  \]
- This assumption is in theory only perfectly correct if daily changes are normally distributed and independent.
The Model-Building Approach

The main alternative to historical simulation is to make assumptions about the probability distributions of the return on the market variables and calculate the probability distribution of the change in the value of the portfolio analytically.

This is known as the model building approach or the variance-covariance approach.
Daily Volatilities

- In option pricing we measure volatility “per year”
- In VaR calculations we measure volatility “per day”

\[ \sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{252}} \]
Daily Volatility continued

- Theoretically, $\sigma_{\text{day}}$ is the standard deviation of the continuously compounded return in one day.
- In practice we assume that it is the standard deviation of the percentage change in one day.
Microsoft Example  (page 502)

- We have a position worth $10 million in Microsoft shares
- The volatility of Microsoft is 2% per day (about 32% per year)
- We use $N=10$ and $X=99$
Microsoft Example continued

The standard deviation of the change in the portfolio in 1 day is $200,000

Assume that the expected change is zero (OK for short time periods) and the probability distribution of the change is

The 1-day 99% VaR is

$$200,000 \times 2.326 = \$465,300$$

The 10-day 99% VaR is

$$\sqrt{10} \times 465,300 = 1,471,300$$
Consider a position of $5 million in AT&T
The daily volatility of AT&T is 1% (approx
16% per year)
The 10-day 99% VaR is

$$\sqrt{10 \times 2.326 \times 50,000} = 367,800$$
Now consider a portfolio consisting of both Microsoft and AT&T.

Assume that the returns of AT&T and Microsoft are bivariate normal.

Suppose that the correlation between the returns is 0.3.
A standard result in statistics states that

\[ \sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho \sigma_X \sigma_Y} \]

In this case \( \sigma_X = 200,000 \) and \( \sigma_Y = 50,000 \) and \( \rho = 0.3 \). The standard deviation of the change in the portfolio value in one day is therefore 220,200.
VaR for Portfolio

- The 10-day 99% VaR for the portfolio is
  \[ 220,200 \times \sqrt{10} \times 2.326 = 1,620,100 \]
- The benefits of diversification are
  \[ (1,471,300 + 367,800) - 1,620,100 = 219,000 \]
- What is the incremental effect of the AT&T holding on VaR?
**The Linear Model**

This assumes

- The daily change in the value of a portfolio is linearly related to the daily returns from market variables
- The returns from the market variables are normally distributed
Markowitz Result for Variance of Return on Portfolio

\[
\text{Variance of Portfolio Return} = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} w_i w_j \sigma_i \sigma_j
\]

- \( w_i \) is weight of \( i \)th instrument in portfolio
- \( \sigma_i^2 \) is variance of return on \( i \)th instrument in portfolio
- \( \rho_{ij} \) is correlation between returns of \( i \)th and \( j \)th instruments
VaR Result for Variance of Portfolio Value \( (\alpha_i = w_i P) \)

\[
\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i
\]

\[
\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j
\]

\[
\sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i<j} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j
\]

\( \sigma_i \) is the daily volatility of \( i \)th instrument (i.e., SD of daily return)

\( \sigma_P \) is the SD of the change in the portfolio value per day
Covariance Matrix \((\text{var}_i = \text{cov}_{ii})\)

\[
C = \begin{pmatrix}
\text{var}_1 & \text{cov}_{12} & \text{cov}_{13} & \cdots & \text{cov}_{1n} \\
\text{cov}_{21} & \text{var}_2 & \text{cov}_{23} & \cdots & \text{cov}_{2n} \\
\text{cov}_{31} & \text{cov}_{32} & \text{var}_3 & \cdots & \text{cov}_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\text{cov}_{n1} & \text{cov}_{n2} & \text{cov}_{n3} & \cdots & \text{var}_n
\end{pmatrix}
\]

\[\text{cov}_{ij} = \rho_{ij} \sigma_i \sigma_j\] where \(\sigma_i\) and \(\sigma_j\) are the SDs of the daily returns of variables \(i\) and \(j\), and \(\rho_{ij}\) is the correlation between them.
Alternative Expressions for $\sigma_P^2$

pages 505-506

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov}_{ij} \alpha_i \alpha_j$$

$$\sigma_P^2 = \alpha^T C \alpha$$

where $\alpha$ is the column vector whose $i$th element is $\alpha_i$ and $\alpha^T$ is its transpose.
Alternatives for Handling Interest Rates

- Duration approach: Linear relation between $\Delta P$ and $\Delta y$ but assumes parallel shifts
- Cash flow mapping: Cash flows are mapped to standard maturities and variables are zero-coupon bond prices with the standard maturities
- Principal components analysis: 2 or 3 independent shifts with their own volatilities
When Linear Model Can be Used

- Portfolio of stocks
- Portfolio of bonds
- Forward contract on foreign currency
- Interest-rate swap
The Linear Model and Options

Consider a portfolio of options dependent on a single stock price, $S$. If $\delta$ is the delta of the option, then it is approximately true that

$$\delta \approx \frac{\Delta P}{\Delta S}$$

Define

$$\Delta x = \frac{\Delta S}{S}$$
Linear Model and Options continued (page 508)

Then

$$\Delta P \approx \delta \Delta S = S \delta \Delta x$$

Similarly when there are many underlying market variables

$$\Delta P \approx \sum_i S_i \delta_i \Delta x_i$$

where $\delta_i$ is the delta of the portfolio with respect to the $i$th asset
Example

Consider an investment in options on Microsoft and AT&T. Suppose the stock prices are 120 and 30 respectively and the deltas of the portfolio with respect to the two stock prices are 1,000 and 20,000 respectively.

As an approximation

\[ \Delta P = 120 \times 1,000 \Delta x_1 + 30 \times 20,000 \Delta x_2 \]

where \( \Delta x_1 \) and \( \Delta x_2 \) are the percentage changes in the two stock prices.
But the distribution of the daily return on an option is not normal

The linear model fails to capture skewness in the probability distribution of the portfolio value.
Impact of gamma (Figure 22.4, page 509)

Positive Gamma

Negative Gamma
Translation of Asset Price Change to Price Change for Long Call (Figure 22.5, page 510)
Translation of Asset Price Change to Price Change for Short Call (Figure 22.6, page 510)
Quadratic Model

For a portfolio dependent on a single stock price it is approximately true that

\[ \Delta P = \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2 \]

this becomes

\[ \Delta P = S \delta \Delta x + \frac{1}{2} S^2 \gamma (\Delta x)^2 \]
Quadratic Model continued

With many market variables we get an expression of the form

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j$$

where

$$\delta_i = \frac{\partial P}{\partial S_i}, \quad \gamma_{ij} = \frac{\partial^2 P}{\partial S_i \partial S_j}$$

But this is much more difficult to work with than the linear model
Monte Carlo Simulation (page 511-512)

To calculate VaR using MC simulation we

- Value portfolio today
- Sample once from the multivariate distributions of the $\Delta x_i$
- Use the $\Delta x_i$ to determine market variables at end of one day
- Revalue the portfolio at the end of day
Monte Carlo Simulation continued

- Calculate $\Delta P$
- Repeat many times to build up a probability distribution for $\Delta P$
- VaR is the appropriate fractile of the distribution times square root of $N$
- For example, with 1,000 trial the 1 percentile is the 10th worst case.
**Speeding up Calculations with the Partial Simulation Approach**

- Use the approximate delta/gamma relationship between $\Delta P$ and the $\Delta x_i$ to calculate the change in value of the portfolio.
- This can also be used to speed up the historical simulation approach.
Comparison of Approaches

- Model building approach assumes normal distributions for market variables. It tends to give poor results for low delta portfolios.
- Historical simulation lets historical data determine distributions, but is computationally slower.
Stress Testing

- This involves testing how well a portfolio performs under extreme but plausible market moves

- Scenarios can be generated using
  - Historical data
  - Analyses carried out by economics group
  - Senior management
Back-Testing

Tests how well VaR estimates would have performed in the past

We could ask the question: How often was the actual 1-day loss greater than the 99%/1-day VaR?
Principal Components Analysis for Swap Rates

- The first factor is a roughly parallel shift (90.9% of variance in data explained)
- The second factor is a twist (another 6.8% of variance explained)
- The third factor is a bowing (another 1.3% of variation explained)
The First Three Principal Components (Figure 22.7, page 515)
Standard Deviation of Factor Scores (bp)

<table>
<thead>
<tr>
<th>PC1</th>
<th>PC2</th>
<th>PC3</th>
<th>PC4</th>
<th>.....</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.55</td>
<td>4.77</td>
<td>2.08</td>
<td>1.29</td>
<td>....</td>
</tr>
</tbody>
</table>
Using PCA to Calculate VaR (page 516)

Example: Sensitivity of portfolio to 1 bp rate move ($m)

<table>
<thead>
<tr>
<th></th>
<th>1 yr</th>
<th>2 yr</th>
<th>3 yr</th>
<th>4 yr</th>
<th>5 yr</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+10</td>
<td>+4</td>
<td>-8</td>
<td>-7</td>
<td>+2</td>
</tr>
</tbody>
</table>

Sensitivity to first factor is from factor loadings:

\[ 10 \times 0.372 + 4 \times 0.392 - 8 \times 0.404 - 7 \times 0.394 + 2 \times 0.376 = -0.05 \]

Similarly sensitivity to second factor = \(-3.87\)
Using PCA to calculate VaR

As an approximation

\[ \Delta P = -0.05f_1 - 3.87f_2 \]

The factors are independent in a PCA

The standard deviation of \( \Delta P \) is

\[
\sqrt{0.05^2 \times 17.55^2 + 3.87^2 \times 4.77^2} = 18.48
\]

The 1 day 99% VaR is \( 18.48 \times 2.326 = 42.99 \)