Chapter 4
Interest Rates
Types of Rates

- Treasury rate
- LIBOR
- Fed funds rate
- Repo rate
Treasury Rate

Rate on instrument issued by a government in its own currency
LIBOR

- LIBOR is the rate of interest at which a AA bank can borrow money on an unsecured basis from another bank.

- For 10 currencies and maturities ranging from 1 day to 12 months it is calculated daily by the British Bankers Association from submissions from a number of major banks.

- There have been some suggestions that banks manipulated LIBOR during certain periods. Why would they do this?
The Fed Funds Rate

- Unsecured interbank overnight rate of interest
- Allows banks to adjust the cash (i.e., reserves) on deposit with the Federal Reserve at the end of each day
- The effective fed funds rate is the average rate on brokered transactions
- The central bank may intervene with its own transactions to raise or lower the rate
Repurchase agreement is an agreement where a financial institution that owns securities agrees to sell them for $X$ and buy them back in the future (usually the next day) for a slightly higher price, $Y$

The financial institution obtains a loan.

The rate of interest is calculated from the difference between $X$ and $Y$ and is known as the repo rate.
Measuring Interest Rates

- The compounding frequency used for an interest rate is the unit of measurement.
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers.
Impact of Compounding

When we compound $m$ times per year at rate $R$ an amount $A$ grows to $A(1+R/m)^m$ in one year.

<table>
<thead>
<tr>
<th>Compounding frequency</th>
<th>Value of $100 in one year at 10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual (m=1)</td>
<td>110.00</td>
</tr>
<tr>
<td>Semiannual (m=2)</td>
<td>110.25</td>
</tr>
<tr>
<td>Quarterly (m=4)</td>
<td>110.38</td>
</tr>
<tr>
<td>Monthly (m=12)</td>
<td>110.47</td>
</tr>
<tr>
<td>Weekly (m=52)</td>
<td>110.51</td>
</tr>
<tr>
<td>Daily (m=365)</td>
<td>110.52</td>
</tr>
</tbody>
</table>
Continuous Compounding

(Page 81)

- In the limit as we compound more and more frequently we obtain continuously compounded interest rates.
- $100 grows to $100e^{RT}$ when invested at a continuously compounded rate $R$ for time $T$.
- $100$ received at time $T$ discounts to $100e^{-RT}$ at time zero when the continuously compounded discount rate is $R$. 

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**Conversion Formulas** (Page 81)

Define

- $R_c$: continuously compounded rate
- $R_m$: same rate with compounding $m$ times per year

\[
R_c = m \ln \left(1 + \frac{R_m}{m}\right)
\]

\[
R_m = m \left( e^{R_c/m} - 1 \right)
\]
Examples

- 10% with semiannual compounding is equivalent to $2 \ln(1.05) = 9.758\%$ with continuous compounding.

- 8% with continuous compounding is equivalent to $4(e^{0.08/4} - 1) = 8.08\%$ with quarterly compounding.

- Rates used in option pricing are nearly always expressed with continuous compounding.
Zero Rates

A zero rate (or spot rate), for maturity $T$ is the rate of interest earned on an investment that provides a payoff only at time $T$.
**Example** *(Table 4.2, page 83)*

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Zero rate (cont. comp.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.0</td>
</tr>
<tr>
<td>1.0</td>
<td>5.8</td>
</tr>
<tr>
<td>1.5</td>
<td>6.4</td>
</tr>
<tr>
<td>2.0</td>
<td>6.8</td>
</tr>
</tbody>
</table>
**Bond Pricing**

To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate.

In our example, the theoretical price of a two-year bond providing a 6% coupon semiannually is

\[
3e^{-0.05 \times 0.5} + 3e^{-0.058 \times 1.0} + 3e^{-0.064 \times 1.5} \\
+ 103e^{-0.068 \times 2.0} = 98.39
\]
Bond Yield

The bond yield is the discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond.

Suppose that the market price of the bond in our example equals its theoretical price of 98.39.

The bond yield (continuously compounded) is given by solving

\[ 3e^{-y \times 0.5} + 3e^{-y \times 1.0} + 3e^{-y \times 1.5} + 103e^{-y \times 2.0} = 98.39 \]

to get \( y = 0.0676 \) or 6.76%.
Par Yield

The par yield for a certain maturity is the coupon rate that causes the bond price to equal its face value.

In our example we solve

\[
\frac{c}{2} e^{-0.05 \times 0.5} + \frac{c}{2} e^{-0.058 \times 1.0} + \frac{c}{2} e^{-0.064 \times 1.5} \\
+ \left( 100 + \frac{c}{2} \right) e^{-0.068 \times 2.0} = 100
\]

to get \( c = 6.87 \) (with semiannual compounding)
In general if $m$ is the number of coupon payments per year, $d$ is the present value of $1$ received at maturity and $A$ is the present value of an annuity of $1$ on each coupon date

$$c = \frac{(100 - 100d)m}{A}$$

(in our example, $m = 2$, $d = 0.87284$, and $A = 3.70027$)
Data to Determine Zero Curve
(Table 4.3, page 84)

<table>
<thead>
<tr>
<th>Bond Principal</th>
<th>Time to Maturity (yrs)</th>
<th>Coupon per year ($)</th>
<th>Bond price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.25</td>
<td>0</td>
<td>97.5</td>
</tr>
<tr>
<td>100</td>
<td>0.50</td>
<td>0</td>
<td>94.9</td>
</tr>
<tr>
<td>100</td>
<td>1.00</td>
<td>0</td>
<td>90.0</td>
</tr>
<tr>
<td>100</td>
<td>1.50</td>
<td>8</td>
<td>96.0</td>
</tr>
<tr>
<td>100</td>
<td>2.00</td>
<td>12</td>
<td>101.6</td>
</tr>
</tbody>
</table>

* Half the stated coupon is paid each year
The Bootstrap Method

- An amount 2.5 can be earned on 97.5 during 3 months.
- Because $100 = 97.5e^{0.10127 \times 0.25}$ the 3-month rate is 10.127% with continuous compounding.
- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding.
The Bootstrap Method continued

To calculate the 1.5 year rate we solve

\[ 4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96 \]

to get \( R = 0.10681 \) or 10.681%

Similarly the two-year rate is 10.808%
Zero Curve Calculated from the Data (Figure 4.1, page 86)
Forward Rates

The forward rate is the future zero rate implied by today’s term structure of interest rates
**Formula for Forward Rates**

- Suppose that the zero rates for time periods $T_1$ and $T_2$ are $R_1$ and $R_2$ with both rates continuously compounded.
- The forward rate for the period between times $T_1$ and $T_2$ is
  \[
  \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}
  \]
- This formula is only approximately true when rates are not expressed with continuous compounding.
## Application of the Formula

<table>
<thead>
<tr>
<th>Year ((n))</th>
<th>Zero rate for (n)-year investment ((% \text{ per annum}))</th>
<th>Forward rate for (n)\text{th} year ((% \text{ per annum}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.0</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>4.6</td>
<td>5.8</td>
</tr>
<tr>
<td>4</td>
<td>5.0</td>
<td>6.2</td>
</tr>
<tr>
<td>5</td>
<td>5.5</td>
<td>6.5</td>
</tr>
</tbody>
</table>
**Instantaneous Forward Rate**

The instantaneous forward rate for a maturity $T$ is the forward rate that applies for a very short time period starting at $T$. It is

$$R + T \frac{\partial R}{\partial T}$$

where $R$ is the $T$-year rate.
Upward vs Downward Sloping Yield Curve

- For an upward sloping yield curve:
  Fwd Rate > Zero Rate > Par Yield

- For a downward sloping yield curve:
  Par Yield > Zero Rate > Fwd Rate
A forward rate agreement (FRA) is an OTC agreement that a certain rate will apply to a certain principal during a certain future time period.
Forward Rate Agreement: Key Results

- An FRA is equivalent to an agreement where interest at a predetermined rate, $R_K$, is exchanged for interest at the market rate.
- An FRA can be valued by assuming that the forward LIBOR interest rate, $R_F$, is certain to be realized.
- This means that the value of an FRA is the present value of the difference between the interest that would be paid at interest at rate $R_F$ and the interest that would be paid at rate $R_K$. 

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Valuation Formulas

- If the period to which an FRA applies lasts from $T_1$ to $T_2$, we assume that $R_F$ and $R_K$ are expressed with a compounding frequency corresponding to the length of the period between $T_1$ and $T_2$

- With an interest rate of $R_K$, the interest cash flow is $R_K(T_2 - T_1)$ at time $T_2$

- With an interest rate of $R_F$, the interest cash flow is $R_F(T_2 - T_1)$ at time $T_2$
Valuation Formulas continued

- When the rate $R_K$ will be received on a principal of $L$, the value of the FRA is the present value of
  \[(R_K - R_F)(T_2 - T_1)\]
  received at time $T_2$

- When the rate $R_K$ will be received on a principal of $L$, the value of the FRA is the present value of
  \[(R_F - R_K)(T_2 - T_1)\]
  received at time $T_2$
Example

- An FRA entered into some time ago ensures that a company will receive 4% (s.a.) on $100 million for six months starting in 1 year.
- Forward LIBOR for the period is 5% (s.a.).
- The 1.5 year rate is 4.5% with continuous compounding.
- The value of the FRA (in $ millions) is
  \[
  100 \times (0.04 - 0.05) \times 0.5 \times e^{-0.045 \times 1.5} = -0.467
  \]
Example continued

If the six-month interest rate in one year turns out to be 5.5% (s.a.) there will be a payoff (in $ millions) of

$$100 \times (0.04 - 0.055) \times 0.5 = -0.75$$

in 1.5 years

The transaction might be settled at the one-year point for an equivalent payoff of

$$\frac{-0.75}{1.0275} = -0.730$$
Duration (page 91-94)

Duration of a bond that provides cash flow $c_i$ at time $t_i$ is

$$D = \sum_{i=1}^{n} t_i \left[ \frac{c_i e^{-yt_i}}{B} \right]$$

where $B$ is its price and $y$ is its yield (continuously compounded)
Key Duration Relationship

Duration is important because it leads to the following key relationship between the change in the yield on the bond and the change in its price

\[
\frac{\Delta B}{B} = -D\Delta y
\]
Key Duration Relationship continued

- When the yield \( y \) is expressed with compounding \( m \) times per year

\[
\Delta B = - \frac{BD\Delta y}{1 + y/m}
\]

- The expression

\[
D \frac{1}{1 + y/m}
\]

is referred to as the “modified duration”
Bond Portfolios

- The duration for a bond portfolio is the weighted average duration of the bonds in the portfolio with weights proportional to prices.
- The key duration relationship for a bond portfolio describes the effect of small parallel shifts in the yield curve.
- What exposures remain if duration of a portfolio of assets equals the duration of a portfolio of liabilities?
Convexity

The convexity, $C$, of a bond is defined as

$$C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2} = \sum_{i=1}^{n} c_i t_i^2 e^{-y t_i}$$

This leads to a more accurate relationship

$$\frac{\Delta B}{B} = -D \Delta y + \frac{1}{2} C (\Delta y)^2$$

When used for bond portfolios it allows larger shifts in the yield curve to be considered, but the shifts still have to be parallel
Theories of the Term Structure

Page 96-98

- Expectations Theory: forward rates equal expected future zero rates
- Market Segmentation: short, medium and long rates determined independently of each other
- Liquidity Preference Theory: forward rates higher than expected future zero rates
Liquidity Preference Theory

Suppose that the outlook for rates is flat and you have been offered the following choices

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Deposit rate</th>
<th>Mortgage rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>3%</td>
<td>6%</td>
</tr>
<tr>
<td>5 year</td>
<td>3%</td>
<td>6%</td>
</tr>
</tbody>
</table>

Which would you choose as a depositor? Which for your mortgage?
Liquidity Preference Theory cont

- To match the maturities of borrowers and lenders a bank has to increase long rates above expected future short rates.
- In our example the bank might offer:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Deposit rate</th>
<th>Mortgage rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>3%</td>
<td>6%</td>
</tr>
<tr>
<td>5 year</td>
<td>4%</td>
<td>7%</td>
</tr>
</tbody>
</table>