Winter 14 – AMS225 Homework 1

Due at the beginning of class, Tuesday January 21.

1. Let \( \mathbf{X} = (X_1, X_2, X_3)^T \) be distributed as MVN(\( \mu \), \( \Sigma \)), where

\[
\mu = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}
\]

Which of the following pairs of random variables are independent? Explain.

(a) \( X_1 \) and \( X_2 \).
(b) \( X_1 \) and \( X_3 \).
(c) \( X_2 \) and \( X_3 \).
(d) \( (X_1, X_3) \) and \( X_2 \).
(e) \( X_1 \) and \( X_1 + 3X_2 - 2X_3 \).

2. Suppose \( \mathbf{X} \) is a random vector with mean \( \mu \). Prove that

\[
E(\mathbf{X} - \mu)(\mathbf{X} - \mu)^T = E(\mathbf{XX}^T) - \mu\mu^T.
\]

3. Let \( \mathbf{Z} \in \mathcal{R}^p \) be distributed as MVN(0, \( I_p \)). Given \( \mu \in \mathcal{R}^p \) and a positive definite \( p \times p \) matrix \( \Sigma \), derive methods to transform \( \mathbf{Z} \) into a \( p \)-variate Normal random vector with mean \( \mu \) and covariance \( \Sigma \) using:

(a) the spectral decomposition \( \Sigma = \mathbf{V}\Lambda\mathbf{V}^T \), where \( \mathbf{V}^T \mathbf{V} = \mathbf{I}_p \) and \( \Lambda \) is diagonal.
(b) the Cholesky decomposition \( \Sigma = \mathbf{L}\mathbf{L}^T \), where \( \mathbf{L} \) is lower-triangular.

4. Let \( Y \) be the random variable satisfying \( P(Y = 1) = P(Y = 1) = 1/2 \). Let \( X_1 \sim N(0, 1) \) and \( X_2 = YX_1 \), and let \( \mathbf{X} = (X_1, X_2)^T \). Show that

(a) \( X_2 \sim N(0, 1) \)
(b) however \( \mathbf{X} \) does not have a bivariate Normal distribution.

5. Let \( \mathbf{X} = (X_1, \ldots, X_n)^T \) be a \( n \times p \) matrix whose rows correspond to repeated observations on \( p \) variables. Let

\[
\bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i
\]

denote the \( p \times 1 \) sample mean vector. A common operation in multivariate analysis is to center observations about the sample mean, i.e. to form the centered data matrix \( \tilde{\mathbf{X}} = \mathbf{X} - 1_n\bar{\mathbf{X}}^T \), where \( 1_n \) is the \( n \times 1 \) matrix of ones.

(a) Exhibit an \( n \times n \) projection matrix \( \mathbf{H} \) such that \( \tilde{\mathbf{X}} = (\mathbf{I} - \mathbf{H})\mathbf{X} \). Note: A matrix \( \mathbf{A} \) is a projection matrix iff i. \( \mathbf{A} = \mathbf{A}^T \), and ii. \( \mathbf{AA} = \mathbf{A} \).
(b) Describe the vectors in the subspace that $H$ projects onto, i.e. what do their entries look like?

(c) Show that $I - H$ is also a projection matrix.

(d) Describe the vectors in the subspace that $I - H$ projects onto, i.e. what is the mean of their entries?

6. Use the setup in the previous problem. In addition, assume that $X_1, \ldots, X_n$ are i.i.d. multivariate normal.

(a) Show that the sample mean $\bar{X}$ and the rows of the centered data matrix $\tilde{X}$ are independent.

(b) Show that the sample mean $\bar{X}$ and sample covariance matrix $S_n$ are independent. Hint: Show that $S_n$ is a function of $\tilde{X}$ (be explicit).

7. Prove that if $A$ is a symmetric real-valued matrix, then all eigenvalues of $A$ are real valued. If additionally that $A$ is positive semidefinite, then all eigenvalues are non-negative; and if $A$ is positive definite, then all eigenvalues are positive.

8. Let $X$ be a $p$-variate random vector and $\Sigma$ be its covariance matrix. Prove that $\Sigma$ is positive semidefinite using the basic properties of (univariate) variance. Hint: Recall that a symmetric matrix $A$ is positive semidefinite iff 
\[ x^T A x \geq 0 \]
for all $x$. 

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