Announcements 3/6/19

* Today: C.1 & 6.2

* Monday: EXAM II back
  - cover C.3
  - Quiz on C.1 & 6.2

* Wednesday: Cover C.4

* Monday 3/18/19: EXAM III
  covers C.1, C.2, C.3, C.4
We've learned to **solve right triangles**

- Given one acute angle and one side, we can solve for the remaining acute angle and two sides.
- Given two sides, we can solve for the two acute angles and the remaining side.

**The Keys to Solving Right Triangles:**

- The Pythagorean theorem \( a^2 + b^2 = c^2 \)
- The angles \( A \) and \( B \) are complementary \( \Rightarrow A + B = 90° \)
- The trig functions and their inverse functions.
Today we learn to solve oblique triangles.

An oblique triangle has no right angles.

It turns out, that to solve an oblique triangle we need to know:

- At least one side AND
- any two other measures of the triangle.
  - two angles
  - one angle & one side
  - two sides


1. Two angles & one side  
   (AAS or ASA)

   ![Diagram: Two triangles with specified angles and side lengths]

   Use Law of Sines

2. Two sides & an angle opposite one of them  
   (SSA)

   ![Diagram: Triangle with specified sides and angle]

   This is the tricky case

   SSA
3. Three Sides (SSS)

(a = 30, b = 11, c = 10)

Use Law of Cosines

4. Two Sides & Their included angle (SAS)

(b = 7, c = 10)
The keys to solving oblique triangles

1. The sum of the angles is $180^\circ \rightarrow A + B + C = 180^\circ$

2. The longest side of the triangle is opposite the largest angle.

3. There can be at most one obtuse angle in a triangle.

4. The trig. functions & their inverse functions.


These laws relate the trig functions of the angles $A, B, C$ to the sides $a, b, c$.

**LAW OF SINES**

If $ABC$ is a triangle with sides $a, b$ and $c$, then

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}.$$
Solve the triangle:

1. First we can solve for the missing angle $B$:

$$135^\circ + 10^\circ + B = 180^\circ$$

$$\rightarrow B = 35^\circ$$

2. Now we can use the law of sines to find the missing sides $b$ and $c$:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Plug in known info
\[
\frac{45}{\sin(135^\circ)} = \frac{b}{\sin(35^\circ)} = \frac{c}{\sin(10^\circ)}
\]

* Solve for \( b \):

\[
\frac{45}{\sin(135^\circ)} = \frac{b}{\sin(35^\circ)} \quad \rightarrow \quad b = \frac{45 \sin(35^\circ)}{\sin(135^\circ)} \approx 34.5022
\]

* Solve for \( c \):

\[
\frac{45}{\sin(135^\circ)} = \frac{c}{\sin(10^\circ)} \quad \rightarrow \quad c = \frac{45 \sin(10^\circ)}{\sin(135^\circ)} \approx 11.0509
\]
Example (ASA, You Try)

Solve the triangle:

\[ \triangle ABC \]

\[ \angle A = 35^\circ, \angle B = 40^\circ, C = 10 \]

1. Solve for \( C \), the missing angle:

\[ 35^\circ + 40^\circ + C = 180^\circ \]

\[ \therefore C = 105^\circ \]

2. Now use the Law of Sines to determine the missing sides \( b \) and \( a \):

- Solve for \( a \):

\[ \frac{a}{\sin(35^\circ)} = \frac{10}{\sin(105^\circ)} \]

\[ a = \frac{10 \sin(35^\circ)}{\sin(105^\circ)} \]

\[ \text{§} \]
\[ \frac{b}{\sin(40^\circ)} = \frac{10}{\sin(105^\circ)} \rightarrow b = \frac{10 \sin(40^\circ)}{\sin(105^\circ)} \]

\[ \approx 6.6546 \]
The Tricky Case \((SSA)\)

* If given two sides and an angle opposite one of them three things could happen:

1. There is exactly one triangle having the two given sides & angle.

2. There are two different triangles with the two given sides & angle.

3. There is no triangle which has the two given sides and angle.

Example \((SSA, \text{ one triangle})\)

Given that \(A = 30^{\circ}, a = 8\) and \(b = 5\) find the remaining side and angles of the triangle.
1. Let's sketch the triangle:

2. Since we know \( A \), \( a \) and \( b \) let's use the law of sines to find the missing angle \( B \):

\[
\frac{a}{\sin(A)} = \frac{b}{\sin(B)} \quad \Rightarrow \quad \sin(B) = \frac{b}{a} \cdot \sin(A)
\]

\[
= \frac{5}{8} \cdot \sin(30^\circ)
\]

\[
\sin(B) = \frac{5}{16}
\]

\[
= \frac{5}{16} \cdot \frac{1}{2}
\]

\[
= \frac{5}{16}
\]
There are two "possible" angles $0 < B < 180^\circ$ where sine is $5/10$.

THE TWO "POSSIBLE" ANGLES

**ACUTE**

$B = \sin^{-1}(\frac{5}{10})$

$\approx 18.2099^\circ$

**OBTUSE**

$B = 180^\circ - \sin^{-1}(\frac{5}{10})$

$\approx 161.7901^\circ$

In this case, $B$ is too large...

$30^\circ + 161.7901^\circ > 180^\circ$

**ALL BAD**

The tricky part of using the Law of Sines.
3. Now solve for the missing angle $C$:

$$30^\circ + 18.2099^\circ + C \times 180^\circ$$

$\Rightarrow C \approx 131.7901^\circ$

4. Finally, use the law of sines to find the missing side $c$:

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)} \quad \Rightarrow \quad c = \frac{a \sin(C)}{\sin(A)}$$

$$c \approx \frac{8 \sin(131.7901^\circ)}{\sin(30^\circ)}$$

$$\approx 11.9295$$
Example (SSA, two triangles)

Find two triangles for which $A = 20^\circ$, $a = 6$ and $b = 15$.

1. Sketch the two triangles

Here both possibilities for the angle $B$ (acute & obtuse) can happen. So we get two different triangles.
(2) Use the law of sines to solve the two triangles:

Triangle #1 (B is acute)

- Solve for B

\[ \frac{a}{\sin(A)} = \frac{b}{\sin(B)} \rightarrow \sin(B) = \frac{b}{a} \sin(A) \]

\[ B = \sin^{-1}(0.8551) \] \[ \approx 58.7708^\circ \]

- Solve for C

\[ 20^\circ + 58.7708^\circ + C = 180^\circ \]

\[ C = 101.2292^\circ \]

- Solve for c

\[ \frac{a}{\sin(A)} = \frac{c}{\sin(C)} \rightarrow c = \frac{a \sin(C)}{\sin(A)} \]

\[ c \approx \frac{6 \sin(101.2292^\circ)}{\sin(20^\circ)} \]

\[ \approx 17.2070 \]
Triangle #2 (B is obtuse)

1. Solve for B

$$B = \arcsin(0.8551)$$

$$\approx 121.2292^\circ$$

2. Solve for C (proceed as last case)

3. Solve for C (proceed as last case)
Example (SSA, no triangle)

Show that there is no triangle for which \( A = 60^\circ, \ a = 6, \ b = 14 \).

1. To get an idea of what's going on, let's attempt to sketch such a triangle:

   ![Diagram of a triangle with angles and sides labeled]

   ? a too short.

   Where does a \( \angle \) intersect the base?

2. Now, use the law of sines to show numerically that no such triangle can exist:

   \[
   \frac{a}{\sin(A)} = \frac{b}{\sin(B)} \quad \rightarrow \quad \sin(B) = \frac{b}{a} \sin(A)
   \]

   \[
   = \frac{14}{6} \sin(60^\circ)
   \]

   \[
   = \frac{14}{6} \cdot \frac{\sqrt{3}}{2}
   \]

   \[
   \approx 2.0207
   \]

   Larger than 1

   ALL BAD
The Area of a Triangle

There is a simple formula for the area of a triangle:

\[ \text{Area} = \frac{1}{2} (\text{base})(\text{height}) \]

Let's look at this formula in terms of oblique triangles. (How do we find the height?)
A is acute

\[ \sin(A) = \frac{h}{b} \]

\[ h = b \sin(A) \]

A is obtuse

\[ \frac{h}{b} = \sin(180^\circ - A) \]
\[ = \sin(180^\circ) \cos(A) - \cos(180^\circ) \sin(A) \]
\[ = 0 \cdot \cos(A) - (-1) \sin(A) \]
\[ = (-1) \sin(A) \]
\[ = \sin(A) \]

\[ h = b \sin(A) \]
Area of a Triangle
If ABC is a triangle with sides a, b and c, then
\[
\text{Area} = \frac{1}{2} c b \sin(A)
\]
\[
= \frac{1}{2} c a \sin(B)
\]
\[
= \frac{1}{2} b a \sin(C)
\]
* Need to know two sides & their included angle.

Example (You Try)
Find the area of a triangle having two sides of lengths 24 feet and 18 feet and an included angle of 60°.

1. Sketch the triangle
2. Use a formula to find the area

\[ A = 80^\circ, \ b = 18\text{ ft}, \ c = 24\text{ ft} \]

\[
\text{Area} = \frac{1}{2} (24\text{ ft})(16\text{ ft}) \sin(80^\circ)
\]

\[ \approx 212.7185 \text{ ft}^2 \]

**LET'S MOVE ON TO THE LAW OF COSINES.**

**WE USE THE LAW OF COSINES IN THE CASES**

SSS \& SAS
LAW OF COSINES

If ABC is a triangle with sides a, b, and c, then

\[ a^2 = b^2 + c^2 - 2bc \cos(A) \]
\[ b^2 = a^2 + c^2 - 2ac \cos(B) \]
\[ c^2 = a^2 + b^2 - 2ab \cos(C) \]

* When using Law of Cosines always find the largest Angle ASAP.

Example (SSS)

Find the three angles A, B, C of the triangle whose sides are \( a = 8 \), \( b = 20 \), \( c = 14 \).
1) Sketch the triangle

2) Use the Law of Cosines to find the largest angle.

In this case, since B is the largest angle, we use the Law of Cosines which involves the angle B.

\[ b^2 = a^2 + c^2 - 2ac \cos(B) \]

\[ 20^2 = 8^2 + 14^2 - 2 \cdot 8 \cdot 14 \cos(B) \]

\[ 400 = 64 + 196 - 224 \cos(B) \]

\[ \cos(B) = \frac{400 - 64 - 196}{-224} \approx 1.28, 68.22^\circ \]

\[ B = \cos^{-1}\left(\frac{-140}{-224}\right) \]

\[ = -\frac{140}{224} \rightarrow B = \cos^{-1}\left(\frac{-140}{224}\right) \]
The KEY FACT:

Likewise if:

\[ \cos(B) > 0 \]

then \( B \) is acute. And the other two angles are acute too.

Since \( \cos(B) < 0 \) we knew that \( B \) is obtuse. So the other two angles must be acute!

The two angles which are not largest are also acute.

(Now can easily apply Law of Sines)

3. We now apply the Law of Sines to determine one of the remaining angles.

- Let's find \( C \).

we knew \( b, B \) and \( c \) so let's use...
\[
\frac{b}{\sin(B)} = \frac{c}{\sin(C)}
\]

\[\Rightarrow \sin(C) = \frac{c}{b} \sin(B)\]

\[\approx \frac{14}{20} \sin(128.6822^\circ)\]

\[\approx 0.5464\]

\[\Rightarrow C \approx \sin^{-1}(0.5464) \approx 33.1229^\circ\]

* Find A

\[128.6822^\circ + 33.1229^\circ + A \approx 180^\circ\]

\[\Rightarrow A \approx 18.1949^\circ\]
Example (SAS)

Find the remaining side and angles

\[ b = 9 \]
\[ c = 12 \]
\[ \angle A = 25^\circ \]

1. Let's start by finding the missing side, \( a \), using the law of cosines.

\[ a^2 = b^2 + c^2 - 2bc \cos(A) \]

\[ a^2 = 9^2 + 12^2 - 2(9)(12) \cos(25^\circ) \]

\[ = 81 + 144 - 216 \cos(25^\circ) \]

\[ = 225 - 216 \cos(25^\circ) \]

\[ a = \sqrt{225 - 216 \cos(25^\circ)} \]

\[ a \approx 4.072 \]
Now we know \( C = 12 \) is the longest side. So \( C \) is the largest angle.

(2) Let's use the law of sines to find the missing angles \( B \) and \( C \).

* First we find \( B \) (remember \( B \) must be acute)

\[
\frac{a}{\sin(A)} = \frac{b}{\sin(B)} \quad \rightarrow \quad \sin(B) = \frac{b}{a} \sin(A)
\]

\[
\approx \frac{9}{5.4072} \sin(25^\circ) \approx 0.7034
\]

\( B \approx \sin^{-1}(0.7034) \approx 44.7026^\circ \)

* Now we can find \( C \)

\[
44.7026^\circ + 25^\circ + C \approx 180^\circ
\]

\( \rightarrow \quad C \approx 110.2972^\circ \)
Example

On a small lake you swim from point A to point B at a bearing of N28°E, then to point C at a bearing of N58°W and finally back to point A. If point C lies 800 meters directly north of point A approximate the total distance you swim.

1. Sketch a diagram
2. From the diagram we get the triangle

\[ \triangle ABC \]

\[ \angle A = 50^\circ, \angle B = 50^\circ, \angle C = 80^\circ \]

\[ AB = BC = AC = 6 \text{ m} \]

We want to know the total distance we swam. This is the sum of the three sides of the triangle.

\[ \text{ASA} \]

We use the Law of Sines.