Announcements (2/20/19)

* Today - Quiz #3 back
  - Finish S.3
  - Quiz #4
  - Start and hopefully finish S.4.

* Mon Feb 25th - Start and finish S.5
  - Review for Exam #2

* Wed Feb 27th - Exam #2
  - Section 6.1
We look at one last example before we move on to 5.4.

**Trig. Equations Involving a Multiple of an Angle**

What if we want to find the solutions to

\[ 2 \cos(3\theta) - 1 = 0 \]

We want to solve the equation for \( \theta \).

We first solve the equation for \( 3\theta \), then divide these solutions by 3.
Example

Solve the equation \(2 \cos(3\theta) - 1 = 0\)

\* Isolate \(\cos(3\theta)\) on one side of the equation.

\[2 \cos(3\theta) - 1 = 0 \rightarrow \cos(3\theta) = \frac{1}{2}\]

\* Over \((0, 2\pi)\) the solutions \(3\theta\) are

\[3\theta = \frac{\pi}{3} \quad \text{and} \quad 3\theta = \frac{5\pi}{3}\]

\* Add all integer multiples of \(2\pi\) to each of these solutions

\[3\theta = \frac{\pi}{3} + 2n\pi\quad \text{for} \ n \ \text{an integer}\]

\[3\theta = \frac{5\pi}{3} + 2n\pi\]
Now divide by 3... 

\[ \theta = \frac{\pi}{9} + \frac{2n\pi}{3} \]

\[ \theta = \frac{5\pi}{9} + \frac{2n\pi}{3} \quad \text{for } n \text{ an integer} \]
Example (You Try)

Solve the equation \(3\tan\left(\frac{\theta}{2}\right) + 3 = 0\).

* Another method is to make a substitution...

Let \(x = \frac{\theta}{2}\)

\[3 \tan(x) + 3 = 0\]

Now solve for \(x\) as usual.

\[3 \tan(x) + 3 = 0 \quad \rightarrow \quad \tan(\frac{x}{3}) = -1\]

\(\tan(x)\) has period \(\pi\) so we first find all solutions \(x\) in the interval \([0, \pi]\)
\[ x = \frac{3\pi}{4} \]

*Since \( \tan(x) \) has period \( \pi \), we add all multiples of \( \pi \) to our solution.*

\[ x = \frac{3\pi}{4} + n\pi \quad \text{for } n \text{ an integer}. \]

Solutions to \( 3\tan(x) + 3 = 0 \), but we want solutions \( \theta \)...

Recall back to \( \frac{\theta}{2} \).

\[ \frac{\theta}{2} = \frac{3\pi}{4} + n\pi \quad \text{for } n \text{ an integer} \]

\[ \downarrow \]

Multiply both sides by 2.

\[ \theta = \frac{3\pi}{2} + 2n\pi \quad \text{for } n \text{ an integer} \]
In this section and in 5.5 we will study several trigonometric identities and how to apply them to:

- Determining the exact value (w/o a calculator) of a trig function at certain angles which are not a quadrant angles and not coterminal to any special angles. Such as

\[
\sin\left(\frac{\pi}{12}\right) \quad \text{and} \quad \cos(105^\circ)
\]

- Trig function values given constrained info.
- Verifying trigonometric identities such as

\[
\cos(\theta - \frac{3\pi}{2}) = -\sin(\theta).
\]

- Solving trigonometric equations such as

\[
\sin(5x) + \sin(3x) = 0.
\]
We will basically be doing what we’ve already done in 5.1, 5.2, 5.3 but with more identities at our disposal than just the fundamental identities...

**SUM & DIFFERENCE FORMULAS**

\[
\sin(u + v) = \sin(u)\cos(v) + \cos(u)\sin(v)
\]

\[
\sin(u - v) = \sin(u)\cos(v) - \cos(u)\sin(v)
\]

\[
\cos(u + v) = \cos(u)\cos(v) - \sin(u)\sin(v)
\]

\[
\cos(u - v) = \cos(u)\cos(v) + \sin(u)\sin(v)
\]

\[
\tan(u + v) = \frac{\tan(u) + \tan(v)}{1 - \tan(u)\tan(v)}
\]

\[
\tan(u - v) = \frac{\tan(u) - \tan(v)}{1 + \tan(u)\tan(v)}
\]
Let's see how to use the sum and difference formulas to find the exact values of trig functions at certain angles which are not quadrant angles and not coterminal to special angles.

Example

Find the exact value of $\sin \left( \frac{\pi}{12} \right)$

*$\frac{\pi}{12}$ is not a quadrant angle and not coterminal to any special angle. However, $\frac{\pi}{12}$ is the difference of two special angles

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$$

* Now we can apply the difference formula

$$\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$$
Here we take

\[ u = \frac{\pi}{3} \quad \text{and} \quad v = \frac{\pi}{4} \]

And we get

\[
\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
= \frac{\sqrt{3}\sqrt{2}}{4} - \frac{\sqrt{2}}{4} \\
= \frac{\sqrt{3}\sqrt{2} - \sqrt{2}}{4} \\
= \frac{\sqrt{2}(\sqrt{3} - 1)}{4}
\]
Example (You Try)

Rewrite the expression

\[ \cos(60^\circ) \cos(45^\circ) - \sin(60^\circ) \sin(45^\circ) \]

as the cosine of a single angle.

Hint: Does this look like the right hand side of a certain sum formula?

\[ \cos(u+v) = \cos(u) \cos(v) - \sin(u) \sin(v) \]

where \( u = 60^\circ \) and \( v = 45^\circ \)

\[ \cos(60^\circ) \cos(45^\circ) - \sin(60^\circ) \sin(45^\circ) = \cos(105^\circ) \]
Example (Constrained information)

Given that $\sin(u) = \frac{4}{5}$ where $u$ is in Q I and $\cos(v) = -\frac{12}{13}$ where $v$ is in Q II find the exact value of $\sin(u+v)$.

*We are asked to determine the exact value of $\sin(u+v)$. We should use which formula?

\[
\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)
\]

given \hspace{1cm} given \hspace{1cm} need to find \hspace{1cm} need to find

\[
= \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \cos(u)\sin(v)
\]
\* Use that we are given \( \sin(u) = \frac{4}{5} \) and \( u \) is in QI to find \( \cos(u) \).

\[
\sin(u) = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}.
\]

Pythagorean Theorem

\[
(\text{adj})^2 = 5^2 - 4^2 = 25 - 16 = 9
\]

so \( \text{adj} = 3 \)

\( u \) is in QI so \( \cos(u) > 0 \)

\[
\cos(u) = \frac{3}{5}
\]

\* Use that we are given \( \cos(v) = -\frac{12}{13} \) and \( v \) is in QII to find \( \sin(v) \).

\[
\cos(v) = \frac{-12}{13} = \frac{\text{adj}}{\text{hyp}}
\]

Pythagorean Theorem

\[
(\text{opp})^2 = 13^2 - (-12)^2 = 169 - 144 = 25
\]

\[
\text{opp} = \sqrt{25} = 5
\]
Finally, plug all the info into the sum formula:

\[
\sin(u + v) = \sin(u)\cos(v) + \cos(u)\sin(v)
\]

\[
= \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)
\]

\[
= -\frac{48}{65} + \frac{15}{65}
\]

\[
= -\frac{33}{65}
\]
Example (You Try)

Simplify the expression \( \cos(\theta - \frac{3\pi}{2}) \) using a difference formula.

Use the difference formula:

\[
\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)
\]

with \( u = \theta \) and \( v = \frac{3\pi}{2} \)

\[
\cos(\theta - \frac{3\pi}{2}) = \cos(\theta)\cos(\frac{3\pi}{2}) + \sin(\theta)\sin(\frac{3\pi}{2})
\]

\[
= \cos(\theta) \cdot 0 + \sin(\theta) (-1)
\]

\[
= -\sin(\theta)
\]
Verify the identity

\[
\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan(\theta)}{1 + \tan(\theta)}
\]

& we use the difference formula

\[
\tan(u - v) = \frac{\tan(u) - \tan(v)}{1 + \tan(u) \tan(v)}
\]

with \( u = \frac{\pi}{4} \) and \( v = \theta \)

\[
\tan\left(\frac{\pi}{4} - \theta\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan(\theta)}{1 + \tan\left(\frac{\pi}{4}\right) \tan(\theta)}
\]

\[
= \frac{1 - \tan(\theta)}{1 + \tan(\theta)}
\]
For our last examples let's see how to use sum and/or difference formulas to solve trig equations.

Example

Find all solutions in the interval \([0, 2\pi]\) to the equation

\[\sin\left(x + \frac{\pi}{2}\right) + \sin\left(x - \frac{3\pi}{2}\right) = 1\]

To start, rewrite each term on the left hand side using a sum or difference formula.
\[ \sin(x + \frac{\pi}{2}) = \sin(x) \cos(\frac{\pi}{2}) + \cos(x) \sin(\frac{\pi}{2}) \]
\[ = \sin(x) \cdot 0 + \cos(x) \cdot 1 \]
\[ = \cos(x) \]

\[ \sin(x - \frac{3\pi}{2}) = \sin(x) \cos(\frac{3\pi}{2}) - \cos(x) \sin(\frac{3\pi}{2}) \]
\[ = \sin(x) \cdot 0 - \cos(x) \cdot (-1) \]
\[ = \cos(x) \]

* Now we can rewrite the entire equation and solve.

\[ \cos(x) + \cos(x) = 1 \]

\[ \Rightarrow 2 \cos(x) = 1 \]

\[ \Rightarrow \cos(x) = \frac{1}{2} \]

We want solutions \( x \) in \( [0, 2\pi) \)
\[ x = \frac{\pi}{3} \quad , \quad x = \frac{5\pi}{3} \]