Announcements 2/13/19

* Today Quiz on 4.7, 4.8 & 5.1
  Form In-class, Finish 5.2 & 5.3

* Next Wed. Quiz on 5.2 & 5.3
  & 5.1. In class, cover 5.4.

5.3 homework problems
4.1, 4.3, 6.1, 6.7
will not be on the quiz.
\* You may remember, from algebra, how to rationalize denominators using Conjugates... 

\[
\frac{1}{\sqrt{2} + \sqrt{3}} = \frac{1}{\sqrt{2} + \sqrt{3}} \cdot \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} - \sqrt{3}} \\
= \frac{\sqrt{2} - \sqrt{3}}{(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})} \\
= \frac{\sqrt{2} - \sqrt{3}}{2 - 3} = -\sqrt{3} + \sqrt{2}
\]

\* The two expressions 

\[A + B \quad & \quad A - B\]

are called **Conjugates**. (Here A \& B can be numbers or variable expressions.)

\[(A + B)(A - B) = A^2 - B^2\]
We can use this technique to help us simplify trig. expressions too.

Example

Verify the identity

\[ \csc(x) + \cot(x) = \frac{\sin(x)}{1 - \cos(x)} \]

* Let's start with the right-hand side and multiply the numerator and denominator by \(1 + \cos(x)\) (the conjugate of the denominator).

\[
\frac{\sin(x)}{1 - \cos(x)} = \frac{\sin(x)(1 + \cos(x))}{(1 - \cos(x))(1 + \cos(x))}
\]

\[
= \frac{\sin(x)(1 + \cos(x))}{1 - \cos^2(x)}
\]

\[
= \frac{\sin(x)(1 + \cos(x))}{\sin^2(x)}
\]
\[
\frac{1 + \cos(x)}{\sin(x)} = \frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)} = \csc(x) + \cot(x)
\]
Let's look at one more example of verifying trig identities...

So far we've always started with one side of the equation and converted it to the other side...

Sometimes it can be helpful to work each side of the equation separately to obtain a common expression which is equivalent to both sides...

(This technique is especially useful if you get stuck.)

Example

Verify the identity

$$\frac{1 - \cos(\theta)}{\cos(\theta)} = \frac{\tan^2(\theta)}{1 + \sec(\theta)}$$
* Let's work on the LHS by itself first.

\[
\frac{1 - \cos(t)}{\cos(t)} = \frac{1}{\cos(t)} - \frac{\cos(t)}{\cos(t)} = \sec(t) - 1
\]

* Now, we're kinda stuck so let's work on the RHS...

\[
\frac{\tan^2(t)}{1 + \sec(t)} = \frac{\sec^2(t) - 1}{1 + \sec(t)} = \frac{(\sec(t) - 1)(\sec(t) + 1)}{1 + \sec(t)} = \sec(t) - 1
\]

These two are the same!
Example (You Try)

Verify the identity

\[
\frac{\cot^2(\theta)}{1 + \csc(\theta)} = \frac{1 - \sin(\theta)}{\sin(\theta)}
\]

RHS

\[
\frac{1 - \sin(\theta)}{\sin(\theta)} = \frac{1}{\sin(\theta)} - \frac{\sin(\theta)}{\sin(\theta)}
\]

\[
= \csc(\theta) - 1 \quad \text{(Now what?)}
\]

LHS

\[
\frac{\cot^2(\theta)}{1 + \csc(\theta)} = \frac{\cot^2(\theta) (1 - \csc(\theta))}{(1 + \csc(\theta))(1 - \csc(\theta))}
\]

\[
= \frac{\cot^2(\theta)(1 - \csc(\theta))}{1 - \csc^2(\theta)}
\]
\[
\begin{align*}
\text{cot}^2(\theta) (1 - \csc(\theta)) & \quad \overline{-(\csc^2(\theta) - 1)} \\
\text{cot}^2(\theta) (1 - \csc(\theta)) & \quad \overline{-\text{cot}^2(\theta)} \\
1 - \csc(\theta) & \quad \overline{-1} \\
\csc(\theta) & - 1
\end{align*}
\]
Here are some examples of trigonometric equations:

\[ 2 \sin(x) = 1 \]

\[ \cot(x) \cos^2(x) = 2 \cot(x) \]

\[ 3 \sec^2(x) - 2 \tan^2(x) - 4 = 0 \]

* To solve a trigonometric equation means to find all real values of the variable for which the equation is true.

We use standard algebraic techniques such as:
- Isolation of terms
- Collecting like terms
- Factoring
- Extracting square roots
- Squaring both sides of the equation.
However, since we are dealing with trig functions we will also use what we know about them to help us solve equations:

- The unit circle
- They are periodic
- The fundamental identities
- Inverse trig functions

Example

Solve the equation $2 \cos(x) = \sqrt{3}$

* Start by isolating the trig function on one side of the equation.

$a \cos(x) = \sqrt{3} \quad \rightarrow \quad \cos(x) = \frac{\sqrt{3}}{2}$
* We know from our knowledge of the unit circle that
\[
\cos(x) = \frac{\sqrt{3}}{2}
\]
has exactly two solutions \( x \) in the interval \([0, 2\pi)\)
\[
x = \frac{\pi}{6} \quad \text{and} \quad x = \frac{11\pi}{6}
\]

* However, these are not the only solutions to this equation!

* Since \( \cos(x) \) is periodic with period \( 2\pi \) there are infinitely many other solutions!

---

Add any integer multiple of \( 2\pi \) to our two solutions to obtain all other solutions. This is written as
\[
x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{11\pi}{6} + 2n\pi
\]
for \( n \) an integer.
Let me give you two perspectives of what is going on here...

1. Let's look at the graph of $y = \cos(x)$.

The $x$-values of the points where the line $y = \frac{\sqrt{3}}{2}$ intersects the graph of $y = \cos(x)$ are the solutions to the equation $\cos(x) = \frac{\sqrt{3}}{2}$.
Another way to think about this is the unit circle. Any angle which is coterminal to $\frac{\pi}{6}$ or $\frac{11\pi}{6}$ is also a solution to the equation $\cos(x) = \frac{\sqrt{3}}{2}$.
Now let's look at various types of trig equations and the techniques used to solve them.

**Example (You Try)**

Solve the equation $1 - \cos(x) = \cos(x)$

(Hint: Start by collecting like terms)

$1 - \cos(x) = \cos(x) \rightarrow 1 = \cos(x) + \cos(x)$

$\rightarrow 1 = 2\cos(x)$

Now isolate $\cos(x)$.

$\cos(x) = \frac{1}{2}$

The solutions to $\cos(x) = \frac{1}{2}$ in the interval $[0, 2\pi)$ are

$x = \frac{\pi}{3}$, $\frac{5\pi}{3}$
Since $\cos(x)$ has period $2\pi$, we add integer multiples to our solutions in $[0, 2\pi)$ to get the general solution:

$$x = \frac{\pi}{3} + 2\pi n, \quad \frac{5\pi}{3} + 2\pi n$$

for $n$ an integer.
Example

Solve the equation $3 \tan^2(x) - 1 = 0$

* Isolate $\tan^2(x)$ on one side of the equation and extract square roots...

\[ 3 \tan^2(x) - 1 = 0 \rightarrow \tan^2(x) = \frac{1}{3} \]

\[ \tan(x) = \pm \frac{1}{\sqrt{3}} \]

* Now, $\tan(x)$ has period $\pi$, so we find all solutions to $\tan(x) = \pm \frac{1}{\sqrt{3}}$ in the interval $[0, \pi)$.

\[ \tan(x) = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad x = \frac{\pi}{6} \quad \text{in} \ [0, \pi) \]

\[ \tan(x) = -\frac{1}{\sqrt{3}} \quad \Rightarrow \quad x = \frac{5\pi}{6} \quad \text{in} \ [0, \pi) \]
Now add multiples of $\pi$ to the solutions we found in $[0, \pi)$ to get the general solution:

\[ x = \frac{\pi}{6} + n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + n\pi \]

for $n$ an integer.
Example (You Try)

Solve the equation \( 3 - 4 \sin^2(x) = 0 \)

\[
3 - 4 \sin^2(x) = 0 \quad \rightarrow \quad \sin^2(x) = \frac{3}{4}
\]

\[
\rightarrow \quad \sin(x) = \pm \frac{\sqrt{3}}{2}
\]

\[
\sin(x) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(x) = -\frac{\sqrt{3}}{2}
\]

\[
\begin{align*}
\left\{ & \sin(x) = \frac{\sqrt{3}}{2} \\
\text{in} [0, 2\pi) \end{align*}
\]

\[
\begin{align*}
\left\{ & \sin(x) = -\frac{\sqrt{3}}{2} \\
\text{in} [0, 2\pi) \end{align*}
\]

\[
\begin{align*}
x &= \frac{\pi}{3}, \frac{2\pi}{3} \\
\end{align*}
\]

\[
\begin{align*}
x &= \frac{4\pi}{3}, \frac{5\pi}{3} \\
\end{align*}
\]

Now add integer multiples of \( 2\pi \) to all 4 of these solutions to obtain the general solution:
\[ x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{2\pi}{3} + 2n\pi \]

\[ x = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2n\pi \]

for \( n \) an integer
Example

Solve the equation $\sin^2(x) = 2 \sin(x)$

* Again, we isolate all the trig functions on one side of the equation.

$\sin^2(x) = 2 \sin(x) \quad \Rightarrow \quad \sin^2(x) - 2 \sin(x) = 0$

* Now factor, and use the zero product property:

$\sin^2(x) - 2 \sin(x) = 0 \quad \Rightarrow \quad \sin(x)(\sin(x)-2) = 0$
\[ \sin(x) = 0 \]

\[ \therefore x = 0, \pi \]

\[ \sin(x) = 2 \]

\[ 2 \text{ is not in the range of \( \sin(x) \)} \]

\[ \boxed{\text{No Solutions}} \]

\* Since the period of \( \sin(x) \) is \( 2\pi \), we collect all solutions in the interval \([0, 2\pi]\) and add integer multiples of \( 2\pi \) to them.

\[ x = 0 + 2n\pi = 2n\pi \]

and

\[ x = \pi + 2n\pi \]

for \( n \) an integer.
Example (You Try)

Solve the equation \( \cot(x) \cos^2(x) = 2 \cot(x) \)

\[
\cot(x) \cos^2(x) = 2 \cot(x) \\
\downarrow \\
\cot(x) \cos^2(x) - 2 \cot(x) = 0 \\
\downarrow \\
\cot(x) \left( \cos^2(x) - 2 \right) = 0 \\
\downarrow \\
\cot(x) = 0 \\
\downarrow \\
in \left( 0, \pi \right) \\
\downarrow \\
x = \frac{\pi}{2}
\]

\[
\cos^2(x) - 2 = 0 \\
\downarrow \\
\cos^2(x) = 2 \\
\downarrow \\
\cos(x) = \pm \sqrt{2}
\]
Add integer multiples of $\pi$ to $\frac{\pi}{2}$ to obtain the general solution.

$\sqrt{2} \neq -\sqrt{2}$

Not in the range of $\cos(x)$

No solutions

$X = \frac{\pi}{2} + n\pi$

$n$ is an integer
Equations of Quadratic Type

You will encounter trig. equations of **quadratic type**

For example:

\[
\frac{\text{quadratic in } \cos(x)}{\quad \downarrow \quad} \quad \frac{\text{quadratic in } \tan(x)}{\quad \downarrow \quad}
\]

\[
2\cos^2(x) - \cos(x) - 1 = 0 \\
\tan^2(x) - 3\tan(x) - 2 = 0
\]

Equations of quadratic type are solved by factoring...

**Example** (You Try)

Find all solutions to the equation

\[
2\sin^2(x) - 3\sin(x) + 1 = 0
\]

in the interval \([0, 2\pi)\).
\[ 2 \sin^2(x) - 3 \sin(x) + 1 = 0 \]
\[ \downarrow \]
\[ 2 \sin^2(x) - 2 \sin(x) - \sin(x) + 1 = 0 \]
\[ \downarrow \]
\[ 2 \sin(x)(\sin(x) - 1) - 1(\sin(x) - 1) = 0 \]
\[ \downarrow \]
\[ (2 \sin(x) - 1)(\sin(x) - 1) = 0 \]
\[ \downarrow \]
\[ 2 \sin(x) - 1 = 0 \] 
\[ \downarrow \]
\[ \sin(x) = \frac{1}{2} \]
\[ \downarrow \]
\[ \ln \left[ 0, 2\pi \right) \]
\[ \downarrow \]
\[ x = \frac{\pi}{6}, \frac{5\pi}{6} \]
\[ \sin(x) = 1 \]
\[ \downarrow \]
\[ \ln \left[ 0, 2\pi \right) \]
\[ \downarrow \]
\[ x = \frac{\pi}{2} \]