5.1: Using the Fundamental Identities

* In Section 5.2 we will learn techniques for verifying trigonometric identities.

* In Section 5.3 we will learn techniques for solving trigonometric equations.

* The key to both of these is... Section 5.1

Be able to use the fundamental identities and rules of algebra to rewrite trigonometric expressions.

* As we discussed last class:

YOU NEED TO MEMORIZE ALL OF THE FUNDAMENTAL IDENTITIES
Let's look at examples of how to use the fundamental identities and rules of algebra to rewrite trigonometric expressions...

Example

Simplify the expression

\[ \cos^2(x) \csc(x) - \csc(x) \]

Note: When asked to "simplify" a trigonometric expression, there will usually be more than one acceptable answer. However, a "simplified" expression should appear simpler:

- Fewer terms
- Fewer functions
- Lower powers
- No fractions
\[
\begin{align*}
\cos^2(x) \csc(x) - \csc(x) &= (\cos^2(x) - 1) \csc(x) \\
&= -(1 - \cos^2(x)) \csc(x) \\
&= -\sin^2(x) \csc(x) \\
&= -\sin^2(x) \cdot \frac{1}{\sin(x)} \\
&= -\sin(x) \cdot \frac{1}{\sin(x)} \\
&= -\sin(x)
\end{align*}
\]
Example (You Try)

Simplify the expression

$$\tan^2(x) - \tan^2(x) \sin^2(x)$$

$$= \tan^2(x) (1 - \sin^2(x))$$

$$= \tan^2(x) \cos^2(x)$$

$$= \frac{\sin^2(x)}{\cos^2(x)} \cos^2(x)$$

$$= \sin^2(x)$$

Pythagorean Identity

Quotient Identity
Example

Factor the expression

$\sec^2(\theta) - 1$

* Look for "special factor forms" such as:

- Difference of squares $\rightarrow A^2 - B^2 = (A-B)(A+B)$
- Trinomials $\rightarrow aX^2 + bX + c$

$\sec^2(\theta) - 1 = (\sec(\theta))^2 - 1^2$ (difference of squares)

$= (\sec(\theta) - 1)(\sec(\theta) + 1)$
Example (You Try)

Factor the expression:

\[ 2 \csc^2(\theta) - 7 \csc(\theta) + 6 \]

*Hint: This expression is a trinomial...*

\[
2 \csc^2(\theta) - 7 \csc(\theta) + 6 \\
= 2 \csc^2(\theta) - 4 \csc(\theta) - 3 \csc(\theta) + 6 \\
= 2 \csc(\theta) (\csc(\theta) - 2) - 3 (\csc(\theta) - 2) \\
= (2 \csc(\theta) - 3)(\csc(\theta) - 2)
\]
Example

Simplify the expression

\[ \cos(t) + \tan(t) \sin(t) \]

What to do?

When in doubt, use the fundamental identities to rewrite everything in terms of sine and cosine...

\[ \cos(t) + \tan(t) \sin(t) \]

\[ = \cos(t) + \frac{\sin(t)}{\cos(t)} \sin(t) \]

\[ = \frac{\cos(t)}{1} + \frac{\sin^2(t)}{\cos(t)} \]

\[ = \frac{\cos(t) \cdot \cos(t)}{\cos(t)} + \frac{\sin^2(t)}{\cos(t)} \]

Final LCD
\[
\begin{align*}
\frac{\cos^2(t)}{\cos(t)} + \frac{\sin^2(t)}{\cos(t)} &= \frac{\cos^2(t) + \sin^2(t)}{\cos(t)} \\
&= \frac{1}{\cos(t)} \\
&= \sec(t)
\end{align*}
\]

Pythagorean identity

Reciprocal identity.
Example (You Try)

Perform the addition

\[ \frac{\sin(t)}{1 + \cos(t)} + \frac{\cos(t)}{\sin(t)} \]

and then simplify.

\[ \frac{\sin(t)}{1 + \cos(t)} + \frac{\cos(t)}{\sin(t)} \]

\[ = \frac{\sin(t)}{1 + \cos(t)} \cdot \frac{\sin(t)}{\sin(t)} + \frac{\cos(t)}{\sin(t)} \cdot \frac{(1 + \cos(t))}{(1 + \cos(t))} \]

\[ = \frac{\sin^2(t)}{\sin(t)(1 + \cos(t))} + \frac{\cos(t) + \cos^2(t)}{\sin(t)(1 + \cos(t))} \]

\[ = \frac{\sin^2(t) + \cos^2(t) + \cos(t)}{\sin(t)(1 + \cos(t))} \]

\[ = \frac{\sin(t)(1 + \cos(t)) + \cos(t)}{\sin(t)(1 + \cos(t))} \]
\[ \frac{1 + \cos(\theta)}{\sin(\theta)(1 + \cos(\theta))} \]

Pythagorean Identity

\[ \frac{1}{\sin(\theta)} \]

Reciprocal Identity

\[ \csc(\theta) \]

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Example

Let's rewrite the expression

\[ \frac{\cos^3(t)}{1 - \sin(t)} \]

so that it is not in fractional form.

* From the Pythagorean Identity we know

\[ \cos^2(t) = 1 - \sin^2(t) = (1 - \sin(t))(1 + \sin(t)) \]

\[
\frac{\cos^3(t)}{1 - \sin(t)} = \frac{\cos^3(t) \cdot 1 + \sin(t)}{1 - \sin(t) \cdot 1 + \sin(t)}
\]

\[
= \frac{\cos^3(t)(1 + \sin(t))}{(1 - \sin(t))(1 + \sin(t))}
\]

\[
= \frac{\cos^3(t)(1 + \sin(t))}{1 - \sin^2(t)}
\]
\[
\frac{\cos^3(t)(1 + \sin(t))}{\cos^7(t)} = \cos(t)(1 + \sin(t))
\]
5.2: Verifying Trigonometric Identities

* Remember, the domain of an equation is the set of all real numbers which make sense to plug in to both sides of the equation.

For example: The domain of the equation

\[
\frac{1}{1-x} = x
\]

is... all real numbers except 1.
There are two types of equations:

**Conditional equations**

These equations are only true for some values in the domain of the equation...

For that reason, these are the equations we "solve" (What values in the domain of the equation make the equation true?)

For example: The equation

\[ \sin(x) = 0 \]

is a conditional equation...

- The **domain** of this equation is all real #s

- However, the equation is only true (both sides are actually equal) when

\[ x = n \pi \] for \( n \) an integer
On the other hand there is another type of equation called an identity.

These types of equations are true for all values of the variable which are in the domain of the equation (any value of the variable so that both sides of the equation make sense.)

For example: The equation

\[ \sin^2(x) = 1 - \cos^2(x) \]

Is an identity because it is true for all real #1's #x (all real #1's is the domain of this equation.)
Let's use our knowledge of the fundamental identities and the rules of algebra to verify some trigonometric identities.

**Example**

Verify the identity

$$\frac{\sec^2(\theta) - 1}{\sec^2(\theta)} = \sin^2(\theta)$$

* Always work on one side of the equation at a time.

* Let's start with the more complicated side...

$$\frac{\sec^2(\theta) - 1}{\sec^2(\theta)} = \frac{\tan^2(\theta)}{\sec^2(\theta)}$$

$$= \left( \frac{\sin^2(\theta)}{\cos^2(\theta)} \right) \frac{1}{\cos^2(\theta)}$$

* Pythagorean Identity.

* Quotient Identities.
\[ \frac{\sin^2(t)}{\cos^2(t)} \cdot \frac{\cos^2(t)}{1} = \sin^2(t) \]
Verify the identity

\[
\frac{\sin^2(t) + \cos^2(t)}{\cos^2(t) \sec^2(t)} = 1
\]

\[
\frac{\sin^2(t) + \cos^2(t)}{\cos^2(t) \sec^2(t)} = \frac{1}{\cos^2(t) \sec^2(t)}
\]

\[
= \frac{1}{\cos^2(t) \cdot \frac{1}{\cos^7(t)}}
\]

\[
= \frac{1}{\cos^2(t)}
\]

\[
= 1
\]
Example

Verify the identity

$$2 \csc^2(\beta) = \frac{1}{1 - \cos(\beta)} + \frac{1}{1 + \cos(\beta)}$$

* Again, let's start with the more complicated side (the right hand side).

$$\frac{1}{1 - \cos(\beta)} + \frac{1}{1 + \cos(\beta)}$$

$$= \frac{1 + \cos(\beta)}{(1 - \cos(\beta))(1 + \cos(\beta))} + \frac{1 - \cos(\beta)}{(1 - \cos(\beta))(1 + \cos(\beta))}$$

$$= \frac{1 + \cos(\beta) + 1 - \cos(\beta)}{(1 - \cos(\beta))(1 + \cos(\beta))}$$

$$= \frac{2}{(1 - \cos(\beta))(1 + \cos(\beta))}$$
\[ \frac{2}{1 - \cos^2(\beta)} = \frac{2}{\sin^2(\beta)} = 2 \frac{1}{\sin^2(\beta)} = 2 \csc^2(\beta) \]
Because there are no standard rules to apply in sequence, verifying trigonometric identities can be challenging. However, the challenge is what makes it fun, like a puzzle. The best way to learn is to practice.

Some Tips

1. Only work on one side of the equation at a time. Start with the more complicated side first.

2. Factor expressions, add fractions, create a monornal denominator etc., (Apply rules of algebra.)

3. Look for ways to use the fundamental identities

4. When in doubt convert every term to sines and cosines.

5. Try something! Don't give up! Mistakes lead to insights!
Example (You Try)

Verify the identity

\((\tan^2(x)+1)(\cos^2(x)-1) = -\tan^2(x)\)

\[
\begin{align*}
\text{LHS} & \quad \Downarrow \\
(\tan^2(x)+1)(\cos^2(x)-1) & = \sec^2(x) \cdot (\cos^2(x)-1) \\
& = -\sec^2(x) \cdot (1-\cos^2(x)) \\
& = -\sec^2(x) \cdot \sin^2(x) \\
& = -\frac{1}{\cos^2(x)} \cdot \sin^2(x) \\
& = -\frac{\sin^2(x)}{\cos^2(x)} \\
& = -\tan^2(x) \quad \Leftrightarrow \text{RHS}
\end{align*}
\]
Example (You Try)

Verify the identity:

\[ \tan(x) + \cot(x) = \sec(x) \csc(x) \]

Hint! Convert to sines & cosines.

* Let's try from two different perspectives...

1) Start with left hand side

\[ \tan(x) + \cot(x) = \frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)} \]

\[ = \frac{\sin^2(x)}{\sin(x) \cos(x)} + \frac{\cos^2(x)}{\sin(x) \cos(x)} \]

\[ = \frac{\sin^2(x) + \cos^2(x)}{\sin(x) \cos(x)} \]

\[ = \frac{1}{\sin(x) \cos(x)} \]
\[
\frac{1}{\sin(x) \cos(x)} = \left( \frac{1}{\sin(x)} \right) \left( \frac{1}{\cos(x)} \right) = \csc(x) \sec(x)
\]
#2) Start with the right hand side:

\[ \sec(x) \csc(x) = \left( \frac{1}{\cos(x)} \right) \left( \frac{1}{\sin(x)} \right) \]

\[ = \frac{1}{\cos(x) \sin(x)} \]

\[ = \frac{\cos^2(x) + \sin^2(x)}{\cos(x) \sin(x)} \]

\[ = \frac{\cos^2(x)}{\cos(x) \sin(x)} + \frac{\sin^2(x)}{\cos(x) \sin(x)} \]

\[ = \frac{\cos(x)}{\sin(x)} + \frac{\sin(x)}{\cos(x)} \]

\[ = \cot(x) + \tan(x) \]