* From last class

* Two angles are coterminal if they have the same initial and terminal sides.
You can find an angle coterminal to a given angle $\theta$ by adding or subtracting $2\pi$, one revolution. (If $\theta$ is measured in degrees, then add or subtract $360^\circ$.)

Any angle $\theta$ has coterminal angles $\theta + 2\pi$ and $\theta - 2\pi$ coterminal to $\theta$.

Remember: unless stated otherwise, angles will always be measured in radians.
A more generally, any angle $\theta$ has infinitely many coterminal angles: add any integer multiple of $2\pi$ to $\theta$ to obtain an angle coterminal to $\theta$.

$\theta + 2n\pi \quad$ coterminal to $\theta$ for any integer $n$.

Example (Finding Coterminal Angles)

Find two angles, one positive and one negative, coterminal to $\frac{2\pi}{3}$.

- To obtain a positive angle coterminal to $\frac{2\pi}{3}$, add $2\pi$.

$$\frac{2\pi}{3} + 2\pi = \frac{2\pi}{3} + \frac{6\pi}{3} = \frac{8\pi}{3}$$

coterminal to $\frac{2\pi}{3}$
To obtain a negative angle coterminal to \( \frac{2\pi}{3} \), subtract \( 2\pi \).

\[
\frac{2\pi}{3} - 2\pi = \frac{2\pi}{3} - \frac{6\pi}{3} = \frac{-4\pi}{3}
\]

To see how this works, let's sketch all three angles \( \frac{2\pi}{3}, \frac{8\pi}{3} \), and \( \frac{-4\pi}{3} \) in standard position.
\[
\frac{\pi}{2} < \frac{2\pi}{3} < \pi
\]

in quadrant II
Complimentary & Supplementary Angles

* Two positive angles are complimentary if their sum is $\frac{\pi}{2}$ (or 90°).

* Two positive angles are supplementary if their sum is $\pi$ (or 180°).

Example (Finding Complementary & Supplementary Angles)

@ Find the complement of $\frac{\pi}{6}$

& Subtract $\frac{\pi}{6}$ from $\frac{\pi}{2}$ ...

$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{3\pi}{6} - \frac{\pi}{6}$$

$$= \frac{2\pi}{6}$$  Complimentary to $\frac{\pi}{6}$

$$= \frac{\pi}{3}$$
(b) Find the supplement of \( \frac{5\pi}{7} \).

& Subtract \( \frac{5\pi}{7} \) from \( \pi \)...

\[ \pi - \frac{5\pi}{7} = \frac{7\pi}{7} - \frac{5\pi}{7} = \frac{2\pi}{7} \] supplementary to \( \frac{5\pi}{7} \)

(c) Find the complement of \( \frac{3\pi}{4} \).

& Since \( \frac{3\pi}{4} > \frac{\pi}{2} \) so \( \frac{3\pi}{4} \) has no complement.

Similarly, an angle \( \theta \) which is larger than \( \pi \) has no supplement.
Applications

Arclength

* The radian measure formula \( \theta = \frac{s}{r} \) can be used to find the arclength \( s \) along a circle.

* For a circle of radius \( r \), a central angle \( \theta \) intercepts an arc of length \( s \) given by \( r \theta \) (\( \theta \) must be measured in radians to use this formula.)
Note: If $r = 1$, then $s = \theta$ and the radian measure of $\theta$ equals the arc length $s$. This will be important for us in section 4.12 and beyond.

Example (Finding Arc Length)

A circle has a radius of 27 inches. Find the length of the arc intercepted by a central angle of $160^\circ$.

- Here the central angle is given in degrees, so we must first convert to radians!

\[
160^\circ = 160^\circ \times \frac{\pi}{180^\circ} = \frac{2\pi}{9}
\]

\[
= \frac{\pi}{2.8^\circ} = \frac{8\pi}{9}
\]
Now we can apply the arc length formula

\[ S = r \cdot \theta \]

\[ \Gamma = \frac{8\pi}{9} \text{ inches} \]

\[ = (2\pi \text{ inches}) \cdot \frac{6\pi}{9} \]

\[ = 24\pi \text{ inches} \]

Let's draw a sketch of this just to see what's going on.

\[ S = 24\pi \text{ inches} \]
Area of a Sector of a Circle

* A Sector of a circle is the region bounded by two radii of the circle, and their intercepted arc.

* For a circle of radius $r$, the area $A$ of the sector of the circle with central angle $\theta$ is given by

$$A = \frac{1}{2} r^2 \theta$$

(\(\theta\) must be in radians to use this formula!)
Example (Area of a Sector of a Circle)

A sprinkler sprays water over a distance of 40 feet and rotates through an angle of 150°. Find the area of watered by the sprinkler.

- Let's sketch a diagram first.

Since the central angle, 150°, is in degrees, we must first convert to radians.

\[
150° = 150° \times \frac{\pi}{180°}
\]

\[
= \frac{3\pi}{2} \times \frac{\pi}{180°}
\]

\[
= \frac{3\pi}{60}
\]

\[
= \frac{5\pi}{6}
\]
No we can use the formula for the area of the sector of a circle.

\[ A = \frac{1}{2} r^2 \theta \]

\[ A = \frac{1}{2} (40 \text{ ft})^2 \cdot \frac{5\pi}{6} \]

\[ A = \frac{800 \cdot 5\pi}{6} \text{ ft}^2 \]

\[ A = \frac{2000\pi}{3} \text{ ft}^2 \]

---

Linear & Angular Speed

The formula \( S = r\theta \), for the length of a circular arc, can help us analyze the motion of an object moving at a constant speed along a circular path.

Constant speed = \( \frac{\text{distance travelled}}{\text{time traveling}} \)

"Average" speed
Consider an object moving at a constant speed along a circular path of radius $r$.

If $S$ is the length of the arc travelled in time $t$, then the linear speed $v$ (pronounced "vello") of the object is

$$v = \frac{S}{t}$$

Moreover, if $\theta$ is the angle (in radians) corresponding to the arc length $S$, then the angular speed $\omega$ (Greek letter omega) of the object is

$$\omega = \frac{\theta}{t}$$
Key: Linear speed measures how fast the object moves in terms of distance travelled per unit time. Angular speed measures how fast the angle changes per unit time.

Since arc length is given by $s = r\theta$, we can relate linear speed to angular speed via the formula:

$$v = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t} = r \cdot \omega$$

Linear Speed = (Radius) \times (Angular Speed)

$$v = r \cdot \omega$$
Example (Finding Angular and Linear Speeds)

A circular blade on a saw rotates at 2400 revolutions per minute.

(a) Find the angular speed of the blade in radians per minute.

\[ 1 \text{ revolution} = 2\pi \text{ radians} \]

\[
\frac{2400 \text{ rev}}{\text{ minute}} = \frac{2400 \cdot 2\pi \text{ radians}}{\text{ minute}}
\]

\[ = 4800 \pi \text{ radians/minute} \]
b) If the blade has a radius of 4 inches, find the linear speed of the blade tip in feet per minute.

\[ \omega = (4 \text{ inches}) \left( \frac{4800 \pi}{\text{minute}} \right) \]

\[ = 19200 \pi \text{ inches/minute} \]

\[ = 19200 \pi \text{ inches/minute} \times \frac{1 \text{ ft}}{12 \text{ inches}} \]

\[ = 1600 \pi \text{ ft/min} \]
4.3: Right Triangle Trigonometry

* In this section we introduce the six trigonometric functions from the right triangle perspective. In section 4.2 we will consider the trigonometric functions more generally.

* A **right angle** is an angle whose measure is \( \frac{\pi}{2} \) radians (or 90°).

* A **right triangle** is a triangle in which one of the angles is a right angle.

\[ \alpha + \beta = 90^\circ \]

The two non-right angles are complementary.

**Pythagorean Theorem**

\[ a^2 + b^2 = c^2 \]

Signifies the 90° angle.
* An angle $\theta$ is called an acute angle if its measure is greater than 0°, but less than 90°:

$$0^\circ < \theta < 90^\circ$$

* In a right triangle the two angles which are not 90° are both acute. (And complimentary!)

* Consider a right triangle with one acute angle labeled $\theta$.

\[ \text{"hyp"} \]

\[ \text{Hypotenuse} \]

\[ \text{The side adjacent} \rightarrow \text{"adj"} \]

\[ \text{The side opposite} \rightarrow \text{"opp"} \]
* The three sides of the triangle are

- The hypotenuse: the longest side of the triangle
- The opposite side: the side opposite to $\theta$
- The adjacent side: the side adjacent to $\theta$

* Using the lengths of these three sides, we can form six ratios which define the six trigonometric functions of the acute angle $\theta$: (Remember, now $0 < \theta < 90^\circ$)

\[
\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \quad \rightarrow \quad \text{"Sine theta"}
\]
\[
\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \quad \rightarrow \quad \text{"Cosine theta"}
\]
\[
\tan(\theta) = \frac{\text{opp}}{\text{adj}} \quad \rightarrow \quad \text{"Tangent theta"}
\]
\[ \csc(\theta) = \frac{\text{hyp}}{\text{opp}} \quad \Rightarrow \quad \text{"cosecant theta"} \]

\[ \sec(\theta) = \frac{\text{hyp}}{\text{adj}} \quad \Rightarrow \quad \text{"secant theta"} \]

\[ \cot(\theta) = \frac{\text{adj}}{\text{opp}} \quad \Rightarrow \quad \text{"cotangent theta"} \]

**Notice:** The last three trigonometric functions are the reciprocals of the first three.

**Example (Evaluating Trigonometric Functions)**

Use the triangle

To find the values of the six trigonometric functions of the angle \( \theta \).
To find the value of all six trigonometric functions of θ we need to know the lengths of all three sides of the triangle (in this case we don't know the length of the opposite side).

* The Pythagorean Theorem says: \[ (\text{hypotenuse})^2 = (\text{opposite})^2 + (\text{adjacent})^2. \]

We use this to solve for the length of the opposite side:

\[
5^2 = (\text{opposite})^2 + 3^2
\]

\[
25 - 9 = (\text{opposite})^2
\]

\[
16 = (\text{opposite})^2
\]

\[\text{opposite} = 4\]
Now we can determine the value of all six trigonometric functions of $\theta$:

\[ \sin(\theta) = \frac{4}{5} \]
\[ \cos(\theta) = \frac{3}{5} \]
\[ \tan(\theta) = \frac{4}{3} \]
\[ \csc(\theta) = \frac{5}{4} \]
\[ \sec(\theta) = \frac{5}{3} \]
\[ \cot(\theta) = \frac{3}{4} \]
In the next two examples we consider two very important types of triangles: isoceles right triangles and equilateral triangles to determine the sine, cosine and tangent of the three "special angles" 30°, 45° and 60°.

* An isoceles right triangle is a right triangle in which two sides are the same length. (In our case we will choose them to be length 1.)
In an isosceles right triangle, the two acute angles are both 45°.

We can use the Pythagorean theorem to find the length of the hypotenuse to be

\[ 1^2 + 1^2 = (\text{hypotenuse})^2 \]

\[ \downarrow \]

\[ 2 = (\text{hypotenuse})^2 \]

\[ \downarrow \]

\[ \text{hypotenuse} = \sqrt{2} \]

Now we can determine the values of \( \sin(45^\circ) \), \( \cos(45^\circ) \) and \( \tan(45^\circ) \). (This also tells us the value of the other three trig. functions at the angle 45°, why?)

\[
\sin(45^\circ) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{← rationalize denominator if you like...}
\]

\[
\cos(45^\circ) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}
\]

\[
\tan(45^\circ) = 1
\]
* An **equilateral triangle** is a triangle in which all three sides are equal.

(In our case we will choose them to all be of length 2.)

- In an equilateral triangle all three angles are 60°.
- An equilateral triangle is not a right triangle, but we can form one by bisecting the bottom side.
- By bisecting the bottom side we halved the top angle to 30°.
We can use the Pythagorean theorem to find the length of the missing side to be

\[ a^2 = b^2 + c^2 \]

\[ a - b = b^2 \]

\[ b = \sqrt{3} \]
Now we get:

\[
\begin{align*}
\sin(60^\circ) &= \frac{\sqrt{3}}{2} \\
\cos(60^\circ) &= \frac{1}{2} \\
\tan(60^\circ) &= \sqrt{3}
\end{align*}
\]

\[
\begin{align*}
\sin(30^\circ) &= \frac{1}{2} \\
\cos(30^\circ) &= \frac{\sqrt{3}}{2} \\
\tan(30^\circ) &= \frac{1}{\sqrt{3}}
\end{align*}
\]
Notice that

\[ \sin(30) = \frac{1}{2} = \cos(60) \]

This is not a coincidence, and happens because 30° and 60° are complimentary angles.

* In general, if \( \theta \) is an acute angle, then:

\[ \sin(90 - \theta) = \cos(\theta) \]

\[ \cos(90 - \theta) = \sin(\theta) \]

**WARNING**: When using your calculator to evaluate trigonometric functions of angles, make sure your calculator is set to the correct mode!
Trigonometric Identities

In trigonometry there is a large focus on studying the relationships between the trigonometric functions. (Identities between the functions.)

Reciprocal Identities

\[
\begin{align*}
\csc(\theta) &= \frac{1}{\sin(\theta)} \\
\sec(\theta) &= \frac{1}{\cos(\theta)} \\
\cot(\theta) &= \frac{1}{\tan(\theta)}
\end{align*}
\]
Quotient Identities

\[ \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \]

\[ \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \]

Pythagorean Identities

\[ \sin^2(\theta) + \cos^2(\theta) = 1 \]

\[ 1 + \cot^2(\theta) = \csc^2(\theta) \]

\[ \tan^2(\theta) + 1 = \sec^2(\theta) \]
* Notie here that

- \( \sin^2(t) \), represents the quantity \( (\sin(t))^2 \)
  
  read "Sine Squared theta"

- \( \cos^2(t) \), represents the quantity \( (\cos(t))^2 \)
  
  read "Cosine Squared theta"

And so on...

* Let's see a few ways to apply these trigonometric identities...
Example (Applying Trig. Identities)

Let \( \theta \) be an acute angle such that \( \cos(\theta) = \frac{1}{4} \). Find the value of \( \sin(\theta) \) and \( \tan(\theta) \).

We know \( \cos(\theta) = \frac{1}{4} \). To find \( \sin(\theta) \), we must use an identity that relates \( \sin(\theta) \) to \( \cos(\theta) \).

\[
\cos^2(\theta) + \sin^2(\theta) = 1
\]

\[
\sin^2(\theta) = 1 - \frac{1}{16}
\]

\[
\sin^2(\theta) = \frac{15}{16}
\]
\[ \sqrt{\sin^2(\theta)} = \sqrt{\frac{15}{16}} \]

\[ \sin(\theta) = \frac{\sqrt{15}}{4} \]

- Now we know \( \cos(\theta) = \frac{1}{4} \) and \( \sin(\theta) = \frac{\sqrt{15}}{4} \).

To find \( \tan(\theta) \) we use an identity which relates \( \tan(\theta) \) to both \( \cos(\theta) \) and \( \sin(\theta) \).

\[ \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \]

\[ \tan(\theta) = \frac{\sqrt{15}}{4} \]

\[ \cos(\theta) = \frac{1}{4} \]

\[ \sin(\theta) = \frac{\sqrt{15}}{4} \]

\[ \tan(\theta) = \sqrt{15} \]
Example (Using Trig. Identities)

Use trig. identities to transform the left hand side of the equation into the right hand side of the equation.

\[ \tan(\theta) \csc(\theta) = \sec(\theta) \]

- We start with the left hand side

\[ \tan(\theta) \csc(\theta) \]

- It is usually a good strategy to use the trig. identities to write everything in terms of sine and cosine, and make cancellations if possible.

\[ \tan(\theta) \csc(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{1}{\sin(\theta)} = \frac{1}{\cos(\theta)} \]
We can now apply a trig identity to obtain what we wanted.

\[
\frac{1}{\cos(\theta)} = \sec(\theta)
\]

Let's see how this looks if we work continuously:

\[
\tan(\theta) \csc(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{1}{\sin(\theta)}
\]

\[
= \frac{1}{\cos(\theta)}
\]

\[
= \sec(\theta)
\]
Example (using Trig. Identities)

Use trigonometric identities to convert the left hand side of the equation into the right hand side of the equation.

\[(\csc(\theta)+1)(\csc(\theta)-1) = \cot^2(\theta)\]

* Again we start with the left hand side, using trig. identities to transform the expression while keeping our goal in mind.
\[(csc(\theta) + 1)(csc(\theta) - 1) = csc(\theta) \cdot csc(\theta) - csc(\theta) + csc(\theta) - 1\]

Expand first

\[= (csc(\theta))^2 - 1\]

\[= csc^2(\theta) - 1\]

\[= \cot^2(\theta)\]

Since

\[1 + \cot^2(\theta) = \csc^2(\theta)\]