4.1: Radian & Degree Measure

* An angle is determined by rotating a ray (half-line) about its endpoint.

* The starting position of the ray is called the initial side of the angle. The position after rotation is called the terminal side of the angle.

* The endpoint of the ray is the vertex of the angle.
* It is very useful to introduce a coordinate system when working with angles.

* In the xy-plane, an angle is in **standard position** if the initial side of the angle coincides with the positive x-axis, with vertex at the origin.

* Counter-clockwise rotation generates **positive angles**.

* Clockwise rotation generates **negative angles**.

Direction of rotation is important!!!
Usually angles are labeled with Greek letters such as:

\[ \alpha \quad \beta \quad \gamma \]

or upper case letters such as: A, B, C.
* The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. There are two common measures of angles: **Degree Measure** and **radian measure**.

**Degree Measure** → Degrees are denoted with the symbol °.

One counterclockwise revolution through a full circle is 360°.

1° is equivalent to a counterclockwise rotation of \( \frac{1}{360} \) of a revolution.

- Zero rotation:
  - 0°
\[ \frac{1}{4} \left( 360^\circ \right) = 90^\circ \]

\[ \frac{3}{8} \left( -360^\circ \right) = -135^\circ \]
\[ \frac{13}{12} \times (360^\circ) = \frac{13 \times 360^\circ}{12} \]
\[ = 13 \times 30^\circ \]
\[ = 390^\circ \]

* To measure angles, it is convenient to mark degrees on the circumference of a circle centred at the origin:
* Decimals can be used to denote fractions of degrees such as:

\[ 64.31^\circ \quad \text{or} \quad -18.12^\circ \]

* Historically, fractional parts of degrees were expressed in minutes and seconds using prime (') and double prime (") notation:

\[ 1' = \text{one minute} = \frac{1}{60} \ (1^\circ) \]

\[ 60' = 1^\circ \]

\[ 1" = \text{one second} = \frac{1}{3600} \ (1^\circ) \]

\[ 3600" = 1^\circ \]

* Explored further in the homework.
Radian Measure

- Especially useful in calculus. We will mostly use radian measure.

* Place the vertex of an angle \( \theta \) at the center of a circle of radius \( r \).

![Diagram of radian measure]

\[ S = \text{length of arc} \]

* The initial and terminal sides of \( \theta \) intersect the circle. Let \( S \) denote the length of the arc formed.

Note: An angle whose vertex is at the center of a circle is called a central angle.
* The **radian measure** of the angle $\theta$ is the ratio of the arc length $s$ to the radius $r$.

$$\theta = \frac{s}{r} \text{ radians}$$

* What is 1 radian?

When the arc length $s$ equals the radius $r$, we get an angle of 1 radian.

$$\Rightarrow \text{ when } s = r$$

$$\theta = \frac{s}{r} \text{ radians} = 1 \text{ radian}$$

* What is the radian measure of one counterclockwise revolution?

After one full revolution, the arc length $s$ is equal to the circumference of the circle which is $2\pi r$. So one counterclockwise revolution corresponds to $2\pi$ radians.
\[ \Rightarrow \text{when } S = 2\pi r \]

\[ \theta = \frac{S}{r} \text{ radians} \]

\[ = \frac{2\pi r}{r} \text{ radians} \]

\[ = 2\pi \text{ radians} \]

* We can use this to determine the radian measure of more angles!

- \( \frac{1}{4} \) revolution, counterclockwise:

\[ \theta = \frac{1}{4} (2\pi \text{ rad}) \]

\[ = \frac{\pi}{2} \text{ rad} \]
\[ \frac{1}{2} \text{ revolution, Counterclockwise:} \]

\[ \theta = \frac{1}{2} (2\pi \text{ rad}) = \pi \text{ rad} \]

\[ \frac{3}{4} \text{ revolution, Counterclockwise:} \]

\[ \theta = \frac{3}{4} (2\pi \text{ rad}) = \frac{3\pi}{2} \text{ rad} \]
* Remember that the four quadrants in the $x$-$y$-plane are numbered I, II, III, and IV and are given by

![Diagram of the x-y plane with quadrants I, II, III, and IV]

* When the terminal side of an angle $\theta$ lies in a quadrant we say $\theta$ lies in that quadrant.
Here are the quadrants in which the angles between 0 and 2\pi lie. (Note that the angles \(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\) and 2\pi do not lie in any quadrant.)

\[
\begin{align*}
\text{Quadrant I} & \quad 0 < \theta < \frac{\pi}{2} \\
\text{Quadrant II} & \quad \frac{\pi}{2} < \theta < \pi \\
\text{Quadrant III} & \quad \pi < \theta < \frac{3\pi}{2} \\
\text{Quadrant IV} & \quad \frac{3\pi}{2} < \theta < 2\pi
\end{align*}
\]
Converting Between Degree and Radian Measure

Since 360° and 2π radians both correspond to one full counterclockwise revolution we get the equation

\[ 360^\circ = 2\pi \text{ radians} \]

If we divide both sides by 2 we get

\[ 180^\circ = \pi \text{ radians} \]

From this last equation we get both:

\[ 1^\circ = \frac{\pi}{180} \text{ radians} \]

\[ \frac{180^\circ}{\pi} = 1 \text{ radian} \]
These two equations give us the conversion rules:

1. To convert from degrees to radians, multiply degrees by

\[
\frac{\pi \text{ radians}}{180^\circ} \rightarrow \text{equal to } \frac{\pi}{2}
\]

2. To convert from radians to degrees, multiply radians by

\[
\frac{180^\circ}{\pi \text{ radians}} \rightarrow \text{equal to } \frac{1}{2}
\]
Example (Convert from degrees to radians)

(a) \[ 60^\circ = \frac{60^\circ \cdot \pi \text{ radians}}{360^\circ} = \frac{\pi}{3} \text{ radians} \]

(b) \[ -225^\circ = \frac{-225^\circ \cdot \pi \text{ radians}}{180^\circ} = \frac{-5 \pi}{4} \text{ radians} \]

\[ = \frac{-45 \pi \text{ radians}}{360} = \frac{-9 \cdot 5 \pi \text{ radians}}{9 \cdot 4} = \frac{-5 \pi}{4} \text{ radians} \]
Example (convert from radians to degrees)

a) \( \frac{\pi}{6} \) radians = \( \frac{\pi}{6} \) radians \( \frac{180^\circ}{\pi \text{ radians}} \)

\[ = \frac{180^\circ}{6} \]
\[ = 30^\circ \]

b) \( -\frac{8\pi}{3} \) radians = \( -\frac{8\pi}{3} \) radians \( \frac{180^\circ}{\pi \text{ radians}} \)

\[ = -\frac{8 \cdot 180^\circ}{3} \]
\[ = -8 \cdot 60^\circ \]
\[ = -480^\circ \]
Co-terminal Angles

* Two angles are called **coterminal** if they have the same initial and terminal sides.

\[ \alpha \text{ is coterminal to } \beta \]

(They are not equal in general)

**Note:** If no unit of measurement is specified for an angle we are measuring in radians, sometimes abbreviated “rad”
* You can find an angle coterminal to a given angle \( \theta \) by adding or subtracting \( 2\pi \) (one revolution). If \( \theta \) is measured in degrees, then add or subtract \( 360^\circ \).

* Any given angle \( \theta \) has infinitely many coterminal angles: add any integer multiple of \( 2\pi \) to \( \theta \) to obtain coterminal angles.

If \( n \) is any integer, then

\[
\theta + 2n\pi
\]

is coterminal to \( \theta \). (\( \theta \) measured in radians.)
Example (Finding Coterminal angles)

Find two angles, one positive & one negative, which are coterminal to $\frac{3\pi}{4}$

* Positive coterminal angle

If we add $2\pi$ (one full counterclockwise revolution) to $\frac{3\pi}{4}$ we get a positive angle which is coterminal to $\frac{3\pi}{4}$.

$$\frac{3\pi}{4} + 2\pi = \frac{3\pi}{4} + \frac{8\pi}{4} = \frac{11\pi}{4}$$

Sketch of both angles in standard position
* Negative Coterminial angle

If we subtract $2\pi$ (one full clockwise revolution) from $\frac{3\pi}{4}$, we get a negative angle which is coterminal to $\frac{3\pi}{4}$.

\[
\frac{3\pi}{4} - 2\pi = \frac{3\pi}{4} - \frac{8\pi}{4} = -\frac{5\pi}{4}
\]

Sketch of both angles in standard position.