1.1: Functions

* A function is a correspondence between two sets \( X \) and \( Y \) that associates to each element in \( X \) exactly one element in \( Y \).

**Example:**

\[
\text{CURRENT UCSC STUDENTS} \quad \xrightarrow{f} \quad \text{STUDENT ID NUMBER}
\]

* The set \( X \) is called the **domain** of the function \( f \).

* The set of "outputs" in \( Y \) is called the **range** of the function \( f \).
Example: This correspondence defines a function:

<table>
<thead>
<tr>
<th>Input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td>13</td>
<td>16</td>
</tr>
</tbody>
</table>

Domain = \{1, 2, 3, 4, 5\}

Range = \{4, 7, 10, 13, 16\}

Non-example

<table>
<thead>
<tr>
<th>Input</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>8</td>
<td>6</td>
<td>5</td>
<td>16</td>
</tr>
</tbody>
</table>

This correspondence does not define a function since 2 corresponds to both 6 and 16...

Example: This set of ordered pairs defines a function:

\[ \{ (1, 5), (2, 8), (3, 4), (4, 9) \} \]

The first elements are the inputs (Domain) and the second are the outputs (Range).

Domain = \{1, 2, 3, 4\}

Range = \{4, 5, 8, 9\}
We can also write this function as a mapping:

<table>
<thead>
<tr>
<th>DOMAIN</th>
<th>RANGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

*When we use formulas to express functions whose inputs are real numbers, convention says that the domain is the largest possible set of real numbers for which the formulas "make sense."

Example: The domain of the function defined by

\[ f(x) = \sqrt{3x + 12} \]

is the set of all real numbers \( x \) which satisfy

\[ 3x + 12 \geq 0 \]

\[ \Rightarrow 3x \geq -12 \]

\[ \Rightarrow x \geq -4 \]
So the domain of \( f \) in set notation is:

\[
\text{Domain} = \{ x \mid x \geq -4 \}
\]

or in interval notation we write:

\[
\text{Domain} = [-4, \infty)
\]

Example: What is the domain of the function defined by

\[
y = \frac{x - 1}{x^2 + 3x - 10}
\]

- Is \( x = 3 \) in the domain?

\[
x = 3 \implies y = \frac{3 - 1}{3^2 + 3(3) - 10} = \frac{2}{9} = \frac{2}{9} = \frac{1}{y}
\]

- Is \( x = 2 \) in the domain?

\[
x = 2 \implies y = \frac{2 - 1}{2^2 + 3(2) - 10} = \frac{1}{4 + 6 - 10} = \frac{1}{0}
\]

Noooooo!
So the domain of this function is all real
real #'s x satisfying

\[ x^2 + 3x - 10 \neq 0. \]

\[ \Rightarrow \text{Find all } x \text{ satisfying } x^2 + 3x - 10 = 0 \text{ and throw them out}: \]

\[ x^2 + 3x - 10 = 0 \]

\[ \Rightarrow (x - 2)(x + 5) = 0 \]

\[ \Rightarrow x - 2 = 0 \text{ or } x + 5 = 0 \]

\[ \Rightarrow x = 2 \text{ or } x = -5 \]

Then the domain is

\[ \left\{ x \in \mathbb{R} : x \neq -5, x \neq 2 \right\} \]

or also written

\[ (-\infty, -5) \cup (-5, 2) \cup (2, \infty) \]

\[ \text{Notation: for a function } f:\]

\[ \text{value in domain } = x \text{ input } = \text{ independent variable} \]

\[ \text{corresponding value in range } = y \text{ output } = \text{ dependent variable} \]
1.2: The Graph of A Function

* The graph of a function \( f \) in the \( xy \)-plane consists of all points \((x, y)\) satisfying:

\( x \) is in the domain of \( f \) and \( y = f(x) \).

\[ (x, y) = (x, f(x)) \]

Remember: Given a graph in the \( xy \)-plane, we can determine if we have a function \( y \) of \( x \) using the Vertical Line Test:

* If any vertical line crosses the graph at most once, then the graph defines a function.
Non-example: The graph of a circle in the $xy$-plane does not define a function:

Equation of circle of radius 1

\[ x^2 + y^2 = 1 \]

All points $(x, y)$ that satisfy this lie on the circle of radius 1.

Example: Sketch the graph of the absolute value function $f(x) = |x|$.

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$
**Example:** Sketch the graph of the function

\[ f(x) = -\sqrt{x} \]

**Domain:** \([0, \infty)\)

**Range:** \((-\infty, 0]\)

**Example:** Sketch the graph of the function

\[ f(x) = \sqrt{25 - x^2} \]

**Domain:**

\[ \exists x : -5 \leq x \leq 5 \]

\[ = [-5, 5] \]

**Range:**

\[ \exists y : 0 \leq y \leq 5 \]

\[ = [0, 5] \]

**First take circle of radius 5**

\[ x^2 + y^2 = 25 \]
* A function $f$ is **increasing** on an interval $I$ if $x_1$ and $x_2$ are real numbers in $I$ satisfying $x_1 < x_2$, then $f(x_1) < f(x_2)$.

A function $f$ is **decreasing** on $I$ if $f(x_2) < f(x_1)$.
If \( f \) is constant on \( I \) if \( f(x_1) = f(x_2) \).

* Given the graph of a function be able to determine exactly where the function is increasing, decreasing and/or constant.

* Points on the graph where the graph changes from rising to falling (or vice versa) are called turning points.
If $f$ is increasing on the intervals $(a, b)$ and $(c, \infty)$

- $f$ is decreasing on the intervals $(-\infty, a)$ and $(b, c)$.

* A function $f$ has a local maximum at $x = b$ if there is an open interval $I$ containing $b$ such that if $x$ is in $I$, then $f(x) \leq f(b)$.

* $f$ has a local minimum at $x = c$ if there is an open interval $I$ containing $c$ such that if $x$ is in $I$, then $f(x) \geq f(c)$.
* $f$ has an **Absolute maximum** at $a$ if $x = a$ if $f(x) \leq f(t)$ for all $x$ in the domain of $f$.
If \( f(p) \leq f(x) \) for all \( x \) in the domain of \( f \), then \( f \) has an absolute minimum at a point \( p \) such that the absolute minimum value is \( f(p) \).
Here is a list of functions you should know:

1. Constant functions: \( f(x) = c \) (\( c \) a real #)

   \[ \text{Domain} = (-\infty, \infty) \]
   \[ \text{Range} = \{ c \} \]

2. The identity function: \( f(x) = x \) for all real #s \( x \)

   \[ \text{Domain} = (-\infty, \infty) \]
   \[ \text{Range} = (-\infty, \infty) \]
3) Square function: \( f(x) = x^2 \)

- Domain: \((-\infty, \infty)\)
- Range: \([0, \infty)\)

4) Cube function: \( f(x) = x^3 \)

- Domain: \((-\infty, \infty)\)
- Range: \((-\infty, \infty)\)
5. **Square root function**: \( f(x) = \sqrt{x} \)

- **Domain**: \([0, \infty)\)
- **Range**: \([0, \infty)\)

6. **Cube root function**: \( f(x) = \sqrt[3]{x} \)

- **Domain**: \((-\infty, \infty)\)
- **Range**: \((-\infty, \infty)\)
7. Reciprocal function: \( f(x) = \frac{1}{x} \)

Domain = \((-\infty, 0) \cup (0, \infty)\)

Range = \((-\infty, 0) \cup (0, \infty)\)

8. Piecewise Defined Functions:

Functions with different equations for different parts of domain.

Example:

\[
 f(x) = \begin{cases} 
 x^2 & \text{if } x > 0 \\
 x & \text{if } x < 0 
\end{cases}
\]

Range = \((-\infty, \infty)\)
The absolute value function \( f(x) = |x| \)

is a piecewise defined function:

\[
|x| = \begin{cases} 
  x & \text{if } x > 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

Domain = \((-\infty, \infty)\)

Range = \([0, \infty)\)
In "real world" problems functions need to be constructed from given information pertaining to the problem.

Example: Express the distance from a point \( P = (x, y) \) on the graph of the function \( y = \sqrt{x} \) to the point \( (1, 0) \) as a function of \( x \).

Solution:
1. Draw a diagram and determine what you are trying to find.
Use "previous knowledge" to relate the given information to the unknown quantity.

Distant formula: \(d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}\)

**Example**: Express the area \(A\) of the rectangle that is inscribed in the semicircle of radius 2 centered at the origin as a function of \(x\).

**Solution**:

1. Draw a diagram.
(2) Relate known to unknown:

* Equation of circle of radius $r$:

$$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \pm \sqrt{r^2 - x^2}$$

* Area of a rectangle

$$A(x) = 2xy = 2 \times \sqrt{y - x^2}$$
2.1: Linear Functions & Their Graphs

* A **linear function** is a function of the form
\[ y = f(x) = mx + b, \quad (m, b \text{ real } \neq 0) \]

* The graph of a line function is a line with slope \( m \) and \( y \)-intercept \((0, b)\).

\[ \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \]

\[ \text{Slope } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \]
The domain of a linear function is all real numbers.

The range of a linear function is all real numbers.

A linear function \( f(x) = mx + b \) is increasing if \( m > 0 \) (slope is positive), decreasing if \( m < 0 \) (slope is negative) and constant if \( m = 0 \) (slope is zero).

**Finding a linear function:**

1. **Given the slope \( m \) and \( y \)-intercept \((0, b)\):**
   
   The linear function is \( y = mx + b \) called slope-intercept form of the line.

2. **Given the slope \( m \) and any point \((x_0, y_0)\) on the line:**
   
   We can determine the equation of the line using the point-slope formula.
3. Given two points \((x_1, y_1)\) and \((x_2, y_2)\) on the line, we can determine the equation of the line:

   - First find the slope of the line:
     \[
     m = \frac{y_2 - y_1}{x_2 - x_1}
     \]

   - Use the slope \(m\) and one of the points (either \((x_1, y_1)\) or \((x_2, y_2)\)) to find the equation of the line using the point-slope form.
Example: Find the equation of the line in the xy-plane which contains the points \((-4, 1)\) and \((2, 11)\).

1. Find the slope \(m\) of the line:

\[ m = \frac{11 - 1}{2 + 4} = \frac{10}{6} = \frac{5}{3} \]

2. Use point-slope form to find eqn of the line:

\[ y - 11 = \frac{5}{3} (x - 2) \]

\[(2, 11)\] lies on the line.

\(\text{Line has slope } \frac{5}{3} \implies \)

\[ y = \frac{5}{3} x - 2\left(\frac{5}{3}\right) + 11 = \frac{5}{3} x + \frac{23}{3} \]
2.3: Quadratic Functions & Their Zeroes

* A quadratic function is of the form

\[ f(x) = ax^2 + bx + c \quad (a, b, c \text{ real } \mathbb{R}) \quad \text{and} \quad a \neq 0. \]

* A quadratic equation is an equation equivalent to one of the form

\[ ax^2 + bx + c = 0 \quad \text{Standard form} \]

Note: Given a quadratic function \( f(x) = ax^2 + bx + c \)
we find the zeroes of \( f \) by solving the quadratic equation \( ax^2 + bx + c = 0. \)

Finding the zeroes of a quadratic function

1. Factoring: It may be possible to solve a quadratic equation by factoring.
   (Not all quadratics can be factored.)
Example: Find the zeroes of the quadratic function \( f(x) = 4x^2 - 17x - 15 \) by factoring.

Solution:

(a) Set the quadratic function equal to zero

\[ 4x^2 - 17x - 15 = 0 \]

(b) Factor the left-hand side of the eqn.

* Find two integers whose product is 

\[ (4)(-15) = -60 \]

and whose sum is 

\[ -17 \]

* Sometimes written like this

\[ \begin{array}{c|c}
4 & \times \\
-15 & -17 \\
\hline
-60 & \end{array} \]

* The two integers are:

\[-20 \text{ and } 3\]
* New factor by grouping:

\[ 4x^2 - 17x - 15 = 4x^2 - 20x + 3x - 15 \]
\[ = (4x^2 - 20x) + (3x - 15) \]
\[ = 4x(x - 5) + 3(x - 5) \uparrow \text{distribute property} \]
\[ = (4x + 3)(x - 5) \]

** Now solve the factored equation:

\( (4x + 3)(x - 5) = 0 \)

\( \Rightarrow \) \( 4x + 3 = 0 \) or \( x - 5 = 0 \)

\( \Rightarrow \) \( x = -\frac{3}{4} \) or \( x = 5 \)
(2) Quadratic Formula: If the quadratic equation

$$ax^2 + bx + c = 0$$

Cannot be factored, or can't figure out how to factor it, use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Zeroes of the quadratic function $f(x) = ax^2 + bx + c$

Remember: If the discriminant $b^2 - 4ac < 0$

of the quadratic $ax^2 + bx + c$,
then there are no real solutions (only imaginary).
3. **Square root Method**
   
   Given a quadratic equation you cannot factor, try the following square root method:

   **Example**: Solve the quadratic equation:

   \[(3x - 5)^2 = 8\]

   **Solution**:

   1. Take the square root of both sides of the eqn:
      
      \[\sqrt{(3x - 5)^2} = \sqrt{8} \Rightarrow \pm (3x - 5) = \sqrt{8}\]
      
      \[\Rightarrow 3x - 5 = \pm \sqrt{8}\]

   2. Now solve for \(x\):
      
      \[3x - 5 = \pm \sqrt{8}\]
      
      \[\Rightarrow 3x = 5 \pm \sqrt{8}\]

      \[\Rightarrow x = \frac{5 \pm \sqrt{8}}{3}\]
There is a nice technique for solving equations of quadratic type. The technique involves a substitution.

Example: Find the zeroes of

\[ f(x) = (2x+5)^2 - (2x+5) - 6 \]

Solution:

a) Set the equation equal to zero:

\[ (2x+5)^2 - (2x+5) - 6 = 0 \]

b) Now make a substitution to obtain a new equation that looks more familiar:

\[ \text{Let } t = 2x+5 \]

\[ \Rightarrow t^2 - t - 6 = 0 \]

c) Factor and solve the new equation in \( t \):

\[ (t - 3)(t + 2) = 0 \]

\[ \Rightarrow t - 3 = 0 \text{ or } t + 2 = 0 \]

\[ \Rightarrow t = 3 \text{ or } t = -2 \]
(d) Back-substitute and solve for \( x \):

\[
\begin{align*}
\Rightarrow & \quad 3 = 2x + 5 \quad \text{or} \quad -2 = 2x + 5 \\
\sqrt{\quad} & \\
2x &= 3 - 5 = -2 \quad \Rightarrow \quad 2x &= -2 - 5 = -7 \\
\Rightarrow & \quad x = -1 \quad \Rightarrow \quad x = \frac{-7}{2}
\end{align*}
\]

So the zeros of \( f \) are

\[ x = -1 \quad \text{and} \quad x = \frac{-7}{2} \]
The graph of a quadratic function is a parabola.

* The vertex is the turning point of the parabola.

Note: The vertex corresponds to absolute maximum or absolute minimum depending on whether the parabola opens down or up.
The vertical line passing through the vertex is the line (or axis) of symmetry of the parabola.

There is an equivalent form to the quadratic function \( f(x) = ax^2 + bx + c \) given by

\[
f(x) = a(x-h)^2 + k \leq \quad \text{(Found by completing the square)}
\]

Here the vertex of the parabola and the axis of symmetry can easily be found:

\[
\begin{align*}
\text{Vertex} & \quad (h, k) = \left( \frac{-b}{2a}, f\left(\frac{-b}{2a}\right) \right) \\
\text{Axis of} & \quad \text{Line } x = h = \frac{-b}{2a}
\end{align*}
\]

Using either form \( f(x) = ax^2 + bx + c \) or \( f(x) = a(x-h)^2 + k \) the parabola opens up if \( a > 0 \), and opens down if \( a < 0 \).
Note: Given a quadratic function,
\[ f(x) = ax^2 + bx + c \]

1. If \( a < 0 \), then \( f \) has an absolute maximum at the point \( x = -\frac{b}{2a} \). The maximum value of \( f \) in this case is \( f\left(-\frac{b}{2a}\right) \).

2. If \( a > 0 \), then \( f \) has an absolute minimum at the point \( x = -\frac{b}{2a} \). The minimum value of \( f \) in this case is \( f\left(-\frac{b}{2a}\right) \).
* Find an equation for a quadratic function given its vertex and one other point on its graph (parabola).

Example: Find the equation of the parabola with vertex \((2,5)\) that passes through the point \((6, -3)\).

Solution:

(a) We know the vertex is \((2,5)\). So the equation of the parabola is of the form

\[f(x) = a(x-2)^2 + 5\]  \(\quad (f(x) = a(x-h)^2 + k)\)

(b) We need to find \(a\). But we know another point on the parabola is \((6, -3)\).

So substitute the coordinates of this point into our equation and solve for \(a\).

\[ -3 = a(6-2)^2 + 5 \]
\[ -3 = a(4)^2 + 5 \]
\[ -3 = 16a + 5 \]
\[ -8 = 16a \]
\[ a = -\frac{1}{2} \]
Example: sketch the graph of \( f(x) = 2x^2 + 8x - 10 \)

Solution:

\( \circ \) Since \( a = 2 > 0 \), we know the parabola opens up.

\( \circ \) The parabola's vertex is at \( x = \frac{-b}{2a} = \frac{-8}{2 \cdot 2} = \frac{-8}{4} = -2 \)

\( \Rightarrow \) \( x = -2 \) \( \Rightarrow \) \( f(-2) = 2(-2)^2 + 8(-2) - 10 \)

\( = 2(4) - 16 - 10 \)

\( = 8 - 16 - 10 = -18 \)

So the vertex of the parabola is at the point \(( -2, -18)\).

\( \circ \) Find the \( x \) and \( y \) intercepts:

- \( y \) intercept: \( \Rightarrow \) when \( x = 0 \) \( \Rightarrow \) \( f(0) = -10 \)
- \( x \) intercepts: Find the zeroes of \( f(x) \)
Set \( f(x) = 0 \) and factor:

\[
2x^2 + 8x - 10 = 0
\]

\[
\Rightarrow 2(x^2 + 4x - 5) = 0
\]

\[
\Rightarrow x^2 + 4x - 5 = 0
\]

\[
\Rightarrow (x + 5)(x - 1) = 0 \Rightarrow x = -5 \text{ or } x = 1
\]
2.5: Inequalities Involving Quadratic Functions

* There is a technique for solving an inequality involving a quadratic function $ax^2 + bx + c$.

1. Find all $x$ such that

$$ax^2 + bx + c \geq 0 \tag{or \geq 0}$$
$$ax^2 + bx + c \leq 0 \tag{or \leq 0}$$
$$ax^2 + bx + c < 0 \tag{or < 0}$$

Example: Solve $x^2 + 2x - 15 \geq 0$

Solution:

1. Factor the quadratic:

$$\Rightarrow x^2 + 2x - 15 = (x+5)(x-3) = 0$$

2. The zeroes of the quadratic break the real number line into three intervals:

$(-\infty, -5)$  $(-5, 3)$  $(3, \infty)$

When $x = -5$ or $x = 3$

$x^2 + 2x - 15 \geq 0$

So these are solutions to the inequality.
2) Choose a test point from each interval and use them to determine if the function is positive or negative on that interval.

<table>
<thead>
<tr>
<th>Interval</th>
<th>(-∞, -5)</th>
<th>(-5, 3)</th>
<th>(3, ∞)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test point</td>
<td>-6</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Sign</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
(-6)^2 + 2(-6) - 15 &= -15 \\
(0)^2 + 2(0) - 15 &= -15 \\
(4)^2 + 2(4) - 15 &= -15
\end{align*}
\]

3) Now we know over which intervals (values of x) \( x^2 + 2x - 15 > 0 \).

\[
\Rightarrow \left( (-\infty, -5] \cup [3, \infty) \right)
\]
In "real world" problems quadratic functions are often used to solve optimization problems. These are problems in which we need to find the minimum or maximum of a quantity.

Example: A rancher wants to build a rectangular corral along a (straight) river, so she will only need to build three sides. If she has three hundred feet of fencing material find the dimensions of the corral that will maximize the area.

Solution:

1. Draw a diagram. Decide what you are trying to find and label knowns/unknowns.
2. Use previous knowledge and information in problem to write the Area as a function of $x$:

$$A(x) = x \left( \frac{800 - x}{2} \right)$$

but we know $x + 2y = 300$

$$\Rightarrow 2y = 300 - x$$

Subtract $\frac{300 - x}{2}$ from $x + 2y = 300$

Solve for $x$

3. Maximize $A(x)$ to find the dimensions ($x$ and $y$) which give the maximum area of the corral.

(use what we know about parabolas)

$$A(x) = \Theta \frac{1}{2} x^2 + 150x$$

quadratic function that opens down!

So $A(x)$ obtains maximum value at

$$x = \frac{-b}{2a} = \frac{-150}{2(-\frac{1}{2})} = \frac{-150}{-1} = 150$$

$$\Rightarrow 150$$

if $x = 150$, then $y = \frac{800 - 150}{2} = \frac{650}{2} = 325$
The dimensions which give the coral the maximum area are 150 x 75

Assignments this week

* Orientation Homework due by tomorrow (6/27/18) by 11:59 pm

* Chapter 1 & 2 quiz will open Friday (6/29/18) at 12:00 am and close Saturday (6/30/18) at 11:59 pm. Approximately 5 questions, will have one attempt per question and about 30 mins to complete once you start.

* Chapter 4, Chp 2, Chp 3 homework due by Monday 7/2/18 at 11:59 pm.