1. Consider the rational function: \( f(x) = \frac{x^2 - 4x - 5}{x^2 + 5x + 4} \)

(a) (8 points) Find the function values \( f(-1) \) and \( f(5) \) or state that they do not exist.

**SOLUTION:**

\[
 f(-1) = \frac{(-1)^2 - 4(-1) - 5}{(-1)^2 + 5(-1) + 4} = \frac{1 + 4 - 5}{1 - 5 + 4} = \frac{0}{0} \quad \text{(Division by zero: \( f(-1) \) does not exist)}
\]

\[
 f(5) = \frac{(5)^2 - 4(5) - 5}{(5)^2 + 5(5) + 4} = \frac{25 - 20 - 5}{25 + 25 + 4} = \frac{0}{49} = 0
\]

(b) (8 points) Find the domain of \( f \). Write your answer in interval notation.

**SOLUTION:** Setting the denominator of \( f \) equal to zero we find

\[
x^2 + 5x + 4 = 0 \quad \Rightarrow \quad (x + 4)(x + 1) = 0 \quad \Rightarrow \quad x = -4, \ x = -1
\]

So we exclude \( x = -4 \) and \( x = -1 \) from the set of all real numbers to obtain the domain of \( f \)

\[(-\infty, -4) \cup (-4, -1) \cup (-1, \infty)\]

(c) (8 points) Simplify \( f(x) \) or state that it cannot be simplified.

**SOLUTION:**

\[
f(x) = \frac{x^2 - 4x - 5}{x^2 + 5x + 4}
\]

\[
= \frac{(x - 5)(x + 1)}{(x + 4)(x + 1)}
\]

\[
= \frac{x - 5}{x + 4}
\]
2. (a) (6 points) Find the LCD: \(\frac{3x}{x^2 - 4}, \frac{5x^2 + 1}{x^2 + x - 2}\) and \(\frac{x + 9}{x^2 - 4x + 4}\)

**SOLUTION:** Factor each denominator completely:

\[ \begin{align*}
    x^2 - 4 &= (x + 2)(x - 2) \\
    x^2 + x - 2 &= (x + 2)(x - 1) \\
    x^2 - 4x + 4 &= (x + 2)^2
\end{align*} \]

So the LCD of the three rational expressions is:

\[ (x - 1)(x - 2)(x + 2)^2 \]

(b) (6 points) Add: \(\frac{3x}{x^2 + x - 2} + \frac{2}{x^2 - 4x + 3}\)

**SOLUTION:**

\[
\begin{align*}
\frac{3x}{x^2 + x - 2} + \frac{2}{x^2 - 4x + 3} &= \frac{3x}{(x + 2)(x - 1)} + \frac{2}{(x - 3)(x - 1)} \\
&= \frac{3x(x - 3)}{(x - 3)(x - 1)(x + 2)} + \frac{2(x + 2)}{(x - 3)(x - 1)(x + 2)} \\
&= \frac{3x(x - 3) + 2(x + 2)}{(x - 3)(x - 1)(x + 2)} \\
&= \frac{3x^2 - 7x + 4}{(x - 3)(x - 1)(x + 2)} \\
&= \frac{(3x - 4)(x - 1)}{(x - 3)(x - 1)(x + 2)} \\
&= \frac{3x - 4}{(x - 3)(x + 2)}
\end{align*}
\]
(c) (6 points) Divide: \[ \frac{6x + 2}{x^2 - 1} \div \frac{3x^2 + x}{1 - x} \]

**SOLUTION:**

\[
\frac{6x + 2}{x^2 - 1} \div \frac{3x^2 + x}{1 - x} = \frac{6x + 2}{x^2 - 1} \cdot \frac{1 - x}{3x^2 + x}
\]

\[
= \frac{2(3x + 1)}{(x + 1)(x - 1)} \cdot \frac{-(x - 1)}{x(3x + 1)}
\]

\[
= \frac{-2(3x + 1)(x - 1)}{x(x - 1)(x + 1)(3x + 1)}
\]

\[
= \frac{-2}{x(x + 1)}
\]

(d) (8 points) Simplify: \[ \frac{x - \frac{1}{2x + 1}}{1 - \frac{1}{2x + 1}} \]

See next page for solution
SOLUTION:

\[
\frac{x - \frac{1}{2x + 1}}{1 - \frac{x}{2x + 1}} = \frac{x(2x + 1) - 1}{2x + 1 - x} - \frac{1}{2x + 1}
\]

= \frac{2x^2 + x - 1}{x + 1}

\[
= \frac{2x^2 + x - 1}{2x + 1} \cdot \frac{2x + 1}{x + 1}
\]

= \frac{(2x - 1)(x + 1)}{2x + 1} \cdot \frac{2x + 1}{x + 1}

= 2x - 1
3. (a) (10 points) Carry out the long division: \( 2x + 3 \) \( \overline{2x^3 + 13x^2 + 9x - 6} \)

\[
\begin{array}{r|ccccc}
& x^2 & + 5x & - 3 \\
\hline
2x + 3 & 2x^3 & + 13x^2 & + 9x & - 6 \\
& - 2x^3 & - 3x^2 & & & \\
\hline
& 10x^2 & + 9x & & & \\
& - 10x^2 & - 15x & & & \\
\hline
& & & - 6x & - 6 & \\
& & & 6x & + 9 & \\
\hline
& & & & 3 & \\
\end{array}
\]

SOLUTION:

(b) (4 points) State the dividend, the divisor, the quotient and remainder of this long division.

\[
\begin{align*}
\text{Dividend:} & \quad 2x^3 + 13x^2 + 9x - 6 \\
\text{Divisor:} & \quad 2x + 3 \\
\text{Quotient:} & \quad x^2 + 5x - 3 \\
\text{Remainder:} & \quad 3
\end{align*}
\]
4. (a) (8 points) Solve: \( \frac{1}{x - 1} + \frac{1}{x + 1} = \frac{2}{x^2 - 1} \)

**SOLUTION:**

\[
\frac{1}{x - 1} + \frac{1}{x + 1} = \frac{2}{x^2 - 1}
\]

\[
\Rightarrow \frac{1}{x - 1} + \frac{1}{x + 1} = \frac{2}{(x + 1)(x - 1)} \quad \text{← restrictions: } x \neq -1 \text{ and } x \neq 1
\]

\[
\Rightarrow (x + 1)(x - 1) \left( \frac{1}{x - 1} + \frac{1}{x + 1} \right) = (x + 1)(x - 1) \left( \frac{2}{(x + 1)(x - 1)} \right)
\]

\[
\Rightarrow \frac{(x + 1)(x - 1)}{x - 1} + \frac{(x + 1)(x - 1)}{x + 1} = 2
\]

\[
\Rightarrow (x + 1) + (x - 1) = 2
\]

\[
\Rightarrow 2x = 2
\]

\[
\Rightarrow x = 1
\]

However, our restrictions tell us that \( x \neq 1 \), so there are no solutions.

(b) (10 points) Park rangers introduce 50 elk to a wildlife preserve. The function

\[
f(t) = \frac{250(3t + 5)}{t + 25}
\]

models the elk population, \( f(t) \), after \( t \) years. How many years will it take for the population to increase to 125 elk?

See next page for solution
**SOLUTION:** We set \( f(t) \) equal to 125 and solve the resulting equation:

\[
125 = \frac{250(3t + 5)}{t + 25} \quad \text{← restrictions: } t \neq -25
\]

\[
\Rightarrow 125(t + 25) = \left( \frac{250(3t + 5)}{t + 25} \right) (t + 25)
\]

\[
\Rightarrow 125(t + 25) = 250(3t + 5)
\]

\[
\Rightarrow 125t + 3125 = 750t + 1250
\]

\[
\Rightarrow 1875 = 625t
\]

\[
\Rightarrow t = 3
\]

It will take 3 years for the elk population to reach 125.
5. (a) (8 points) Solve the formula for $a$: \[ R = \frac{as}{a + s} \]

\[
\begin{align*}
R &= \frac{as}{a + s} \\
\Rightarrow R(a + s) &= \left(\frac{as}{a+s}\right)(a+s) \\
\Rightarrow R(a + s) &= as \\
\Rightarrow Ra + Rs &= as \\
\Rightarrow Ra - as &= -Rs \\
\Rightarrow a(R - s) &= -Rs \\
\Rightarrow a &= \frac{-Rs}{R - s}
\end{align*}
\]

(b) (10 points) A passenger train can travel 240 miles in the same amount of time it takes a freight train to travel 160 miles. If the average speed of the freight train is 20 miles per hour slower than the average speed of the passenger train, find the average speed of both trains.

See next page for solution
SOLUTION:

Let $x = \text{speed of freight train} \Rightarrow x + 20 = \text{speed of passenger train}$

<table>
<thead>
<tr>
<th></th>
<th>distance</th>
<th>speed</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>passenger train</td>
<td>240 miles</td>
<td>$x + 20$ m.p.h.</td>
<td>$\frac{240}{x + 20}$ hours</td>
</tr>
<tr>
<td>freight train</td>
<td>160 miles</td>
<td>$x$ m.p.h.</td>
<td>$\frac{160}{x}$ hours</td>
</tr>
</tbody>
</table>

Since it takes the same amount of time for the two trains to travel the specified distances we obtain the following equation to solve:

$$\frac{240}{x + 20} = \frac{160}{x} \quad \leftarrow \text{restrictions: } x \neq -20 \text{ and } x \neq 0$$

$$\Rightarrow x(x + 20) \left( \frac{240}{x + 20} \right) = x(x + 20) \left( \frac{160}{x} \right)$$

$$\Rightarrow 240x = 160(x + 20)$$

$$\Rightarrow 240x = 160x + 3200$$

$$\Rightarrow 80x = 3200$$

$$\Rightarrow x = 40$$

The freight trains speed is 40 m.p.h. and the passenger trains speed is 60 m.p.h.
(c) (Extra Credit) The length of a violin string is inversely proportional to the frequency of its vibrations. If a violin string 16-inches long vibrates at a frequency of 320 cycles per second, at what frequency does a 20-inch string vibrate?

**SOLUTION:** First solve for $k$, the constant of proportionality:

$$16 = \frac{k}{320} \Rightarrow 16 \cdot 320 = k \Rightarrow k = 5120$$

Now find the frequency. A 20-inch violin string vibrates at:

$$20 = \frac{5,120}{\text{frequency}} \Rightarrow \text{frequency} = \frac{5,120}{20} \Rightarrow \text{frequency} = 256.$$  

A 20-inch violin string vibrates at a frequency of 256 cycles per second.