6.1- Rational Expressions and Functions: Multiplying and Dividing

★ A rational expression consists of a polynomial divided by a (nonzero) polynomial. Here are some examples of rational expressions:

\[
\frac{120x}{100 - x}, \quad \frac{2x + 1}{2x^2 - x - 1} \quad \text{and} \quad \frac{3x^2 + 12xy - 15y^2}{6x^3 - 6xy^2}.
\]

A rational function is a function defined by a formula that is a rational expression.

GOAL(S): Be able to evaluate rational functions, find the domain of a rational function, simplify rational expressions and multiply and divide rational expressions.

Example (Evaluate Rational Functions)

The rational function

\[ f(x) = \frac{120x}{100 - x} \]

models the cost, \( f(x) \), in thousands of dollars, to remove \( x\% \) of the pollutants that a city has discharged into a lake. Find and interpret the following (write your descriptions in complete sentences):

a. \( f(20) \) 

b. \( f(80) \)

★ The domain of a rational function is the set of all real numbers except those for which the denominator is zero. (In some cases, though they make sense algebraically, more values may need to be discarded, if they do not make sense in the context of the function...)

Example Continued (Finding the Domain of a Rational Function)

Find the domain of the rational function

\[ f(x) = \frac{120x}{100 - x}. \]

Does the domain indicate that the city can clean up the lake completely?

Example (Finding the Domain of a Rational Function)

Find the domain of each of the following:

a. \( f(x) = \frac{2x + 1}{2x^2 - x - 1} \) 

b. \( g(x) = \frac{2x}{x^2 + 1} \)
A rational expression is simplified (or reduced to lowest terms) if its numerator and denominator have no common factors (other than 1 or -1).

**Example** (Simplifying Rational Expressions)

Simplify each of the following:

a. \( \frac{x^2 + 4x + 3}{x + 1} \)

b. \( \frac{x^2 - 7x - 18}{2x^2 + 3x - 2} \)

We will always assume that a simplified rational expression is equal to the original rational expression for all real numbers except those for which either denominator is zero.

**Simplifying Rational Expressions**

1. Factor both the numerator and denominator completely.
2. Divide out the factors common to the numerator and denominator.

**Example** (Simplifying Rational Expressions)

Simplify: \( \frac{4x^2 - 25}{3(5 - 2x)} \)

**Multiplying Rational Expressions**

1. Factor all numerators and denominators completely.
2. Multiply the factors in the numerators and multiply the remaining factors in the denominators.
3. Divide out the numerator and denominator by common factors.

**Example** (Multiplying Rational Expressions)

Multiply: \( \frac{x + 3}{x - 4} \cdot \frac{x^2 - 2x - 8}{x^2 - 9} \)
★ Just like dividing two rational numbers, we find the **quotient of two rational expressions** by inverting the divisor and then multiplying.

**Example (Dividing Rational Expressions)**

Divide each of the following:

a. \((4x^2 - 25) \div \frac{2x + 5}{14}\)

b. \(\frac{x^2 - 5x + 6}{2x} \div \frac{x^2 - 5x - 6}{x^2 - 3x}\)

**Dividing Rational Expressions**

1. Factor all numerators and denominators completely.
2. Invert the divisor and multiply.