11.1 Sequences & Summation Notation

* This material will only be on the final exam as extra credit.

* Sequences of it's "appear" everywhere in nature. For example, The **Fibonacci Sequence** is an infinite sequence of numbers:

\[ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots \]

The first two Fibonacci numbers are 1, after that each Fibonacci number is the sum of the previous two.

\[ 2 = 1 + 1 \]
\[ 3 = 2 + 1 \]
\[ 5 = 3 + 2 \]
\[ 8 = 5 + 3 \]
\[ 13 = 8 + 5 \]

Let's visualize the Fibonacci sequence...
The side lengths are successive Fibonacci #s...
* Spirals such as this one appear in many places in nature, from spiral galaxies to Mollusk shells. (Called logarithmic spirals)

* The Fibonacci sequence appears in other places in nature too: Arrangements of leaves on a stem, the fruitlets of a pineapple, the flowering of an artichoke, and the uncurling arrangement in a pine cone. For more see the Wikipedia page "Fibonacci Numbers".

Inspired by "Majereeeen" on GIPHY
We can think of the Fibonacci sequence as a function $f$. The terms of the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

are the outputs of $f$. The domain of $f$ is the set of positive integers.

\[\text{Domain: } 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ \ldots\]

\[\text{Range: } 1 \ 1 \ 2 \ 3 \ 5 \ 8 \ 13 \ 21 \ \ldots\]

\[\begin{array}{cccccccc}
& f(1) = 1 & , & f(2) = 1 & , & f(3) = 2 & , & f(4) = 3 \\
& f(5) = 5 & , & f(6) = 8 & , & f(7) = 13 & , & f(8) = 21 \\
\end{array}\]
Rather than the usual function notation, the letter $a$ with a subscript is used to represent function values or a sequence.

\[ a_1, a_2, a_3, a_4, a_5, a_6, \ldots \]

The subscripts make up the domain of the sequence and they identify the location of a term in the sequence.

The notation $a_n$ is used to represent the $n$th term, or general term, of a sequence.

We represent the entire sequence as $\{a_n\}$.\[\]
Sequences

An infinite sequence $\{a_n\}$ is a function whose domain is the set of positive integers. The function values $a_1, a_2, a_3, a_4, a_5, \ldots$ are the terms of the sequence.

* Sequences whose domain consists only of a finite # of positive integers are called finite sequences.
Example (Write the Terms of a Sequence)

Write the first four terms of each sequence from their general (or nth term).

a) \( a_n = 3n + 4 \)

b) \( b_n = \frac{(-1)^n}{3^n - 1} \)

a) To find the first four terms of the sequence whose general term is \( a_n = 3n + 4 \), we replace \( n \) in the formula with the positive integers 1, 2, 3, 4.

\[
\begin{align*}
    n=1 & : a_1 = 3(1) + 4 = 7 \\
    n=2 & : a_2 = 3(2) + 4 = 10 \\
    n=3 & : a_3 = 3(3) + 4 = 13 \\
    n=4 & : a_4 = 3(4) + 4 = 16
\end{align*}
\]
So the first few terms of the sequence \( \frac{3n+4}{2n^2} \) are 7, 10, 13, 16.

We can also write the sequence \( \frac{3n+4}{2n^2} \) as

\[
7, 10, 13, 16, \ldots, 3n+4, \ldots
\]

1st term, 2nd term, 3rd term, 4th term, \ldots, nth term.
b) We do the same for the sequence $\sum b_n z^n$ whose general term is

$$b_n = \frac{(-1)^n}{3^n - 1}$$

\begin{align*}
\text{For } n=1, \quad b_1 &= \frac{(-1)^1}{3^1 - 1} = -\frac{1}{2} \\
\text{For } n=2, \quad b_2 &= \frac{(-1)^2}{3^2 - 1} = \frac{1}{8} \\
\text{For } n=3, \quad b_3 &= \frac{(-1)^3}{3^3 - 1} = -\frac{1}{24} \\
\text{For } n=4, \quad b_4 &= \frac{(-1)^4}{3^4 - 1} = \frac{1}{80}
\end{align*}
Factorial notation

For any positive integer \( n \), \( n! \) (read "n factorial") is defined as

\[
n! = (n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)
\]

Zero factorial is defined to be 1.

\[
0! = 1.
\]

Example (Computing Factorials)

Compute

\( a) \ 5! \quad \text{b) } \ (2\cdot3)! \)

The product of the first \( n \) positive integers
a) \(5! = (5)(5-1)(5-2)(5-3)(5-4)\)

\[= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad \text{product of first 5 positive integers}\]

\[= 120\]

b) \((2\cdot3)! = (6)!\)

\[= (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \quad \text{product of first 6 positive integers}\]

\[= 720\]
What if we want to sum the first, say, seven terms of the Fibonacci sequence \( \{a_n\} \):

\[
1 + 1 + 2 + 3 + 5 + 8 + 13 = 33
\]

\[
a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7
\]

A much more compact and convenient way to write this sum is in Sommation Notation:

\[
\sum_{i=1}^{7} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7
\]

\[
\begin{array}{cccccccc}
i = 1 & \uparrow & i = 2 & \uparrow & i = 3 & \uparrow & i = 4 & \uparrow \uparrow \uparrow \\
i = 5 & \uparrow & i = 6 & \uparrow & i = 7 & \end{array}
\]
We read this expression as

"The sum as i goes from 1 to 7 of q_i."

Think of the symbol $\sum$ as instructions to add up the terms of a sequence.

**Summation Notation**

The sum of the first $n$ terms of a sequence is written in summation notation as

$$\sum_{i=1}^{n} q_i = q_1 + q_2 + q_3 + \ldots + q_n$$

- $i$ is the index of summation.
- $n$ is the lower limit of summation.
- $n$ is the upper limit of summation.
→ Any letter can be used as the index of summation (usually i, j or k).

The index of summation just keeps track of the terms we are summing...

→ The lower limit of summation tells us which term of the sequence to start the sum with. (Can start at any positive integer, not just 1)

→ The upper limit of summation tells us which term of the sequence to end the sum with.

\[
\sum_{i=6}^{11} a_i = a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11}
\]

Start with 6th term

Take all consecutive terms between 6th and 11th

end with 11th term
When we write out a sum which is given in summation notation, we are expanding the sum.

Example (Expanding Sums)

Expand and Evaluate the sums

a) \[\sum_{i=1}^{6} (i^2 + 1)\]

b) \[\sum_{k=4}^{7} 2^k - 1\]

c) \[\sum_{i=1}^{5} 3\]

a) To expand the sum \[\sum_{i=1}^{6} (i^2 + 1)\] we replace \( i \) in the expression \( i^2 + 1 \) with the consecutive integers 1, 2, 3, 4, 5, and 6 and add...
\[
\sum_{i=1}^{6} (i^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) + (5^2 + 1) + (6^2 + 1)
\]

\[
= 2 + 5 + 10 + 17 + 26 + 37.
\]

\[
= 97
\]
6) \[ \sum_{k=4}^{7} (2^k - 1) = (2^4 - 1) + (2^5 - 1) + (2^6 - 1) + (2^7 - 1) \]

\[ = (16 - 1) + (32 - 1) + (64 - 1) + (128 - 1) \]

\[ = 15 + 31 + 63 + 127 \]

\[ = 236 \]
c) To find the sum $\sum_{i=1}^{5} 3$ we first notice that each term of the sum is 3. The summation notation indicates that we must sum the first five terms ($i$ goes from 1 to 5) of the sequence in which every term is 3...

$$\sum_{i=1}^{5} 3 = 3 + 3 + 3 + 3 + 3$$

$$a_1 + a_2 + a_3 + a_4 + a_5$$

all are 3's

= 15
A lets go "backwards"...

Example (Write a Sum in Summation Notation)

Express each sum in summation notation:

a) \(1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3\)

b) \(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n-1}}\)

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a) \(1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3\)

There are 7 terms of the form \(i^3\) as \(i\) goes from 1 to 7.

\[
1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = \sum_{i=1}^{7} i^3
\]
b) \[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^{n-1}}\]

\[
\begin{align*}
1 &= \frac{1}{3^0} = \frac{1}{3} \times \frac{1}{3^1} \\
\frac{1}{3} &= \frac{1}{3^1} = \frac{1}{3^2} \times \frac{1}{3^2} \\
\frac{1}{9} &= \frac{1}{3^2} = \frac{1}{3^3} \times \frac{1}{3^3} \\
\frac{1}{27} &= \frac{1}{3^3} = \frac{1}{3^4} \times \frac{1}{3^4} \\
\text{This is the } n\text{th term of the sequence.}
\end{align*}
\]

There are \(n\)-terms each of the form \[\frac{1}{3^{i-1}}\] for \(i\) from 1 to \(n\).

\[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^{n-1}} = \sum_{i=1}^{n} \frac{1}{3^{i-1}}\]