The Final Exam is in our usual room on Monday December 10th, from 1:45-3:45pm. No notes will be allowed during the exam. A scientific calculator will be allowed. The exam is cumulative and will cover:

- **Chapter 4**: 4.1, 4.2
- **Chapter 6**: 6.1, 6.2, 6.3, 6.4, 6.6, 6.7
- **Chapter 7**: 7.1, 7.2, 7.3, 7.4, 7.5, 7.6
- **Chapter 9**: 9.1, 9.2, 9.3, 9.4, 9.5, 9.6
- **Chapter 10**: 10.1

Make sure you are comfortable with all of the assigned homework problems from these sections. (The homework for chapters 9 & 10 will be due at the beginning of the exam.)

**Formulas to be Supplied on the Exam**

\[
A = P \left(1 + \frac{r}{n}\right)^{nt}
\]

\[
A = Pe^{rt}
\]

\[
f(f^{-1}(x)) = x \quad \& \quad f^{-1}(f(x)) = x
\]

\[
\log_b(M \cdot N) = \log_b(M) + \log_b(N)
\]

\[
\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)
\]

\[
\log_b(M^p) = p \cdot \log_b(M)
\]

\[
\log_b(M) = \frac{\log_a(M)}{\log_a(b)}
\]

\[
A = A_0e^{kt}
\]

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)
\]

\[
(x - h)^2 + (y - k)^2 = r^2
\]

\[
x^2 + y^2 + Dx + Ey + F = 0
\]
Topics and Breakdown of the Exam

• Chapter 4:
  – (15 points) Solve an absolute value inequality, graph the solution set on the number line and state the solution set in interval notation.

• Chapter 6:
  – (5 points) Find the domain of a rational function.
  – (5 points) Determine the LCD of two or more rational expressions.
  – (10 points) Simplify a complex rational expression.
  – (10 points) Solve a rational equation.

• Chapter 7:
  – (5 points) Find the domain of a square root function.
  – (5 points) Convert between radical notation and rational exponents.
  – (10 points) Rationalize denominators and simplify radical expressions.
  – (10 points) Solve a radical equation.

• Chapter 9:
  – (5 points) Match an exponential function with its graph.
  – (10 points) Apply compound interest formulas.
  – (10 points) Use the switch and solve method to find the inverse of a one-to-one function.
  – (5 points) Apply the horizontal line test.
  – (5 points) Sketch the graph of $f^{-1}$ using the graph of $f$.
  – (10 points) Use basic properties of logarithms and evaluate logarithms.
  – (5 points) Find the domain of a logarithmic function.
  – (10 points) Solve a logarithmic equation.
  – (15 points) Determine an exponential growth or decay model from given data, and use the model to make predictions.

• Chapter 10:
  – (5 points) Find the distance between two points in the plane.
  – (5 points) Find the midpoint of the line segment joining two points in the plane.
  – (10 points) Use the standard form of a circles equation to graph the circle.
  – (10 points) Convert the general form of a circles equation to the standard form.

★ I rewrote the review of Completeing The Square on page 28 so it is more clear.
A One of algebra's applications is to predict the behavior of variables (modeling).

One very important and useful way this is done is using exponential growth and exponential decay models.

In exponential growth and exponential decay models we are concerned with quantities that grow or decay at a rate which is directly proportional to their size.

\[
\begin{align*}
\text{• Populations grow more and more rapidly, the larger they are...} \\
\text{• Organic material decays more and more slowly, the less of it there is...}
\end{align*}
\]
Goal(s): Be able to use exponential growth and exponential decay models to predict future behavior.

* Two variable quantities are said to be directly proportional if their ratio is constant. This constant ratio is known as the Constant of Proportionality.

* For example, if an object moves at a constant speed, then the distance travelled by an object and the time the object is in motion are directly proportional.

\[
\text{distance} = \text{rate} \cdot \text{time}
\]

\[
\frac{\text{distance}}{\text{time}} = \text{rate}
\]

The rate of speed is the Constant of proportionality.
Now suppose we have a quantity \( A \) that varies with time \( t \). If the rate of change of \( A \) is directly proportional to \( A \), then we can model how \( A \) is changing with an exponential growth or exponential decay model.

\[
\text{Rate of change of } A = \frac{\text{d}A}{\text{d}t} = kA
\]

\( k \) is the constant of proportionality.

**Exponential Growth**

- \( A = \text{Amount at time } t \)
- \( A_0 = \text{Initial amount at } t = 0 \)
- \( k > 0 \)

**Exponential Decay**

- \( A = \text{Amount at time } t \)
- \( A_0 = \text{Initial amount at } t = 0 \)
- \( k < 0 \)
Exponential Growth & Decay Models

The mathematical model for exponential growth & exponential decay is given by

\[ A = A_0 e^{kt} \]

Where:

* \( A \) is the amount of the quantity at time \( t \).
* \( A_0 \) is the initial amount at time \( t=0 \).
* \( K \) is the constant of proportionality.

- If \( K > 0 \), the model describes exponential growth.
- If \( K < 0 \), the model describes exponential decay.
In many applications, one needs to use given data to determine \( K \). After we find \( K \), we can use the model \( A = A_0 e^{kt} \) to make predictions...

**Example (Exponential Growth: Modeling Populations)**

In the year 2000, the population of Africa was 807 million. By 2011 the population had grown to 1,052 million.

a) Use the exponential growth model \( A = A_0 e^{kt} \), where \( t \) is the # of years. After 2000, to find an exponential growth function that models the data.

b) According to the model, by which year will Africa's population reach 2,000 million (2 billion)?
a) We are given:

- The population of Africa is 807 million in the year 2000. (when \( t = 0 \))
- The population of Africa is 1,052 million in the year 2011. (when \( t = 11 \))

We want to use the exponential model

\[ A = A_0 e^{kt} \]

Need to know \( A_0 \) & \( k \) to use (this model)

To model the population, \( A \), in millions of people, of Africa \( t \) years after the year 2000.

What is \( A_0 \)?

\( A_0 = 807 \text{ (million)} \)

Population of Africa in the year 2000 (when \( t = 0 \))

What is \( k \)?

Use \( A_0 = 807 \) and the data given:

\[ A = 1,052 \text{ (millin)} \]

When \( t = 11 \)
So to determine \( k \), we need to solve the exponential equation

\[
A = A_0 e^{kt}
\]

Plug in \( t = 11 \)

\[
1,052 = 807 e^{11k}
\]

\[
\Rightarrow \frac{1,052}{807} = e^{11k}
\]

\[
\Rightarrow \ln \left( \frac{1,052}{807} \right) = ln(e^{11k})
\]

* Solve for \( k \)
\[
\ln \left( \frac{1.052}{807} \right) = 11K \ln(e) \\
\ln \left( \frac{1.052}{807} \right) = 11 \cdot K \\
K = \frac{\ln \left( \frac{1.052}{807} \right)}{11} \\
\]

\[
\approx 0.024 \\
\text{Approximate to 3 decimal places using calculator}
\]

Then the exponential growth model for the population of Africa using the given data is

\[
A = 807e^{0.024t}
\]
b) Question: In what year will the population of Africa reach 2,000 million (2 billion)?

* We use the exponential growth model:

\[ A = 807e^{0.024t} \]

\[ A = 2,000 \]

Now solve this equation for \( t \) ....

\[ 2000 = 807e^{0.024t} \]

\[ \frac{2000}{807} = e^{0.024t} \]

\[ \ln \left( \frac{2000}{807} \right) = \ln \left( e^{0.024t} \right) \]
\[ \ln \left( \frac{2000}{807} \right) = 0.024t \ln(e) \]

\[ \ln \left( \frac{2000}{807} \right) = 0.024t \]

\[ t = \frac{\ln \left( \frac{2000}{807} \right)}{0.024} \]

37.82 years after 2000

The population of Africa is approximately 2,000 Million in the year 2038
Now we'll look at an exponential decay model...

The **half-life** of a substance is the time required for half of a given amount of the substance to decay.

**Example** (Exponential Decay: Carbon-14 Dating)

The age of an artifact or fossil (up to 80,000 years) can be accurately determined by measuring the percentage of Carbon-14 remaining in the fossil or artifact.

a) Use the fact that the half-life of Carbon-14 is 5,715 years to find an exponential decay model for Carbon-14.

b) In 1947, what are known as the Dead Sea Scrolls were found by an Arab Bedouin herdsman. Analysis indicated that the scroll wrappings contained 76% of their original Carbon-14. Estimate the age of the scrolls when they were found in 1947.
a) In this case all we know is:

- The half life of Carbon-14 is 5,715 years.

This means, for any given amount of Carbon-14, it takes 5,715 years for that amount to decay to half of its size.

We want to use the exponential decay model

\[ A = A_0 \cdot e^{kt} \]

to model the amount of Carbon-14 remaining from an initial amount of \( A_0 \) after \( t \) years.

What is \( A_0 \)?

We want to be able to use this model for any initial amount \( A_0 \) of Carbon-14. So we don't need a specific \( A_0 \) in this case...

What is \( k \)?

To use the the half life of Carbon-14...
Given an initial amount $A_0$ of Carbon-14, after 5,715 years we know $\frac{A_0}{2}$ remains... So to determine $K$ we need to solve the exponential equation

$$\frac{A_0}{2} = A_0 \cdot e^{k \cdot 5715}$$

Cancel the $A_0$'s

$$\frac{1}{2} = e^{5715 \cdot K}$$

$$\ln(\frac{1}{2}) = \ln(e^{5715 \cdot K})$$

$$\ln(\frac{1}{2}) = 5715 \cdot K \ln(e)$$

$$\ln(\frac{1}{2}) = 5715 \cdot K$$

$$K = \frac{\ln(\frac{1}{2})}{5715}$$
\[ A = A_0 e^{-0.00012t} \]

The exponential decay model

Can be used to model the decay of Carbon-14.
b) Question: In 1947 the Dead Sea Scrolls contained 76% of their original Carbon-14. How old are the Dead Sea Scrolls, approximately?

We use the exponential decay model:

\[ A = A_0 e^{-0.00012t} \]

When the amount of Carbon-14 is 76% of the initial amount we get:

\[ A = 0.76 A_0 \]

So we need to solve the exponential equation:

\[ 0.76 A_0 = A_0 e^{-0.00012t} \]
\[ 0.76 = e^{-0.00012t} \]

\[ \Rightarrow \ln(0.76) = \ln(e^{-0.00012t}) \]

\[ \Rightarrow \ln(0.76) = -0.00012t \ln(e) \]

\[ \Rightarrow \ln(0.76) = -0.00012t \]

\[ \Rightarrow t = \frac{\ln(0.76)}{-0.00012} \]

\[ t \approx 2283.01 \text{ years old} \] (Approx. to 2 decimal places)

The Dead Sea Scrolls were approximately 2287 years old in 1947.
In this section we develop some analytic geometry (geometry using coordinate systems). We make use of the Pythagorean Theorem:

\[ a^2 + b^2 = c^2 \]

Goals: Find the distance between two points in the plane. Find the midpoint of the line segment joining two points in the plane. Determine the center \& radius of a circle whose equation is given in standard form. Graph circles in the plane. Convert the general form of a circle's equation to standard form.
If we have two points, $a$ and $b$, on the line:

$$a \quad \quad \quad b$$

then the distance between them is $|b - a|$. 

What if we have two points $(x_1, y_1)$ and $(x_2, y_2)$ in the plane:

What is the distance $d$ between $(x_1, y_1)$ and $(x_2, y_2)$?

The distance $d$ between $(x_1, y_1)$ and $(x_2, y_2)$ is given by the Pythagorean Theorem:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$
The Distance Formula

The distance of between the points \((x_1, y_1)\) and \((x_2, y_2)\) in the Cartesian coordinate system is

\[ d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \]

Example (Using The Distance Formula)

Find the distance between the points \((-1, -3)\) and \((2, 3)\). Express the answer in simplified radical form & then round to two decimal places.

\((x_1, y_1) = (-1, -3)\) and \((x_2, y_2) = (2, 3)\)

\[ d = \sqrt{(2-(-1))^2 + (3-(-3))^2} \]

\[ = \sqrt{9 + 36} \]

\[ = \sqrt{45} \]
\[ \sqrt{45} \]

\[ = \sqrt{3^2 \cdot 5} \]

\[ = 3 \sqrt{5} \quad \text{Simplified radical} \]

* Approximate to 2 decimal places

\[ 3 \sqrt{5} \approx 6.71 \]
* If we have two points a and b on the line:

![Diagram showing points a and b on a number line]

The midpoint between them is \( \frac{a + b}{2} \).

The average of the two points (#15)

* What is the midpoint between two points \((x_1, y_1)\) and \((x_2, y_2)\) in the plane?

![Diagram showing points \((x_1, y_1)\) and \((x_2, y_2)\) with midpoints calculated]

What is the midpoint of the line segment joining \((x_1, y_1)\) and \((x_2, y_2)\)?

* So to find the midpoint we need to take the midpoint of each of the coordinates separately.
The Midpoint Formula

The midpoint of the line segment joining the two points \((x_1, y_1) \neq (x_2, y_2)\) in the Cartesian coordinate system is

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

Example (using The Midpoint Formula)

Find the midpoint of the line segment with endpoints \((1, 2)\) and \((7, -3)\)

Plug into formula! \((x_1, y_1) = (1, 2)\)

\[\Rightarrow \quad (x_2, y_2) = (7, -3)\]

\[
\Rightarrow \quad \text{Midpoint} \quad \left( \frac{1 + 7}{2}, \frac{2 + (-3)}{2} \right) = (4, -1/2)
\]
What is a circle?

Circles

A circle is the set of all points in the plane which are equidistant from a fixed point called the center. The distance from the center to any point on the circle is called the radius.

Let's convert the geometry of the circle to an equation in the Cartesian coordinates $x$ and $y$. 
* Place the circle of radius $r$ in the $xy$-plane and label its center $(h,k)$.

* The distance from any point $(x,y)$ on the circle to the center is $r$, the radius. So the distance formula gives us

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

Square both sides

$$r^2 = (x-h)^2 + (y-k)^2$$

* This equation gives an algebraic description of all points $(x,y)$ lying on the circle of radius $r$ and center $(h,k)$. 

\[\text{\square} \]
The Standard Form: (Equation of a Circle)

The standard form of the equation of the circle with center \((h, k)\) and radius \(r\) is

\[(x-h)^2 + (y-k)^2 = r^2\]

Example (The Standard Form) Write the standard form of the equation of the circle with center \((0, 0)\) and radius 4.

\[
\begin{cases}
  h = 0 \\
  k = 0 \\
  r = 4
\end{cases}
\]

Plug into formula \((x-h)^2 + (y-k)^2 = r^2\)

\[x^2 + y^2 = 16\]
Example (The Standard Form) Write the standard form of the equation of the circle with center $(5, -6)$ and radius 10.

\[
\begin{align*}
\begin{cases}
h = 5 \\
k = -6 \\
r = 10
\end{cases} \quad & \Rightarrow \text{Plug into formula} \\
(x - 5)^2 + (y - (-6))^2 &= 10^2 \\
\Rightarrow (x - 5)^2 + (y + 6)^2 &= 100
\end{align*}
\]
Example (Graphing a Circle From Standard Form) 12/3/18

Find the center and radius of the circle whose equation in standard form is

\[(x+3)^2 + (y-1)^2 = 4\]

Center: \((-3, 1)\)

Radius: 2

Don't need to graph the center since it doesn't lie on the circle, but it helps when graphing.
If we expand the standard form of the equation of the circle, and relabel the coefficients (the constants), we get the general form of the equation of a circle.

* For example, the standard form of the equation of a circle of radius 3 and center \((1, 2)\) is

\[(x - 1)^2 + (y - 2)^2 = 9\]

* If we expand this equation, we get the general form:

\[(x^2 - 2x + 1) + (y^2 - 4y + 4) = 9\]

\[x^2 + y^2 - 2x - 4y - 4 = 0\]

General form of equation
Completing The Square

\[ x^2 + bX \]

\[ = x^2 + bX + \frac{b^2}{4} - \frac{b^2}{4} \]

\[ = \left(x^2 + bX + \frac{b^2}{4}\right) - \frac{b^2}{4} \]

\[ = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} \]

- **Add the Square of half the coefficient of X** (half the coefficient of X is \(\frac{b}{2}\), and its square is \(\frac{b^2}{4}\)).
- To avoid changing the expression, we must also subtract \(\frac{b^2}{4}\),...
- Now group these terms together.
- Finally, we notice the grouped terms are a perfect square (hence, the name "completing the square").
\[ \text{The General Form (Equation of a Circle)} \]

The general form of the equation of a circle is

\[ x^2 + y^2 + Dx + Ey + F = 0 \]

We cannot read off the center and radius of the circle from the general form of its equation. We need to rewrite it in standard form... This involves completing the square (twice...)

\[ \text{Example (Convert: General to Standard Form)} \]

Write in standard form:

\[ x^2 + y^2 + 4x - 4y - 1 = 0 \]
* Group together the x terms and the y terms. Move the constants to the other side.

\[ x^2 + y^2 + 4x - 4y - 1 = 0 \]

\[ \Rightarrow (x^2 + 4x) + (y^2 - 4y) = 1 \]

* Complete the square in x

\[ (x^2 + 4x) + (y^2 - 4y) = 1 \]

\[ \Rightarrow (x^2 + 4x + 4) - 4 + (y^2 - 4y) = 1 \]

\[ \Rightarrow (x + 2)^2 - 4 + (y^2 - 4y) = 1 \]
* Complete the square in $y$

$$(x+2)^2 - y + (y^2 - 4y + 4) = 1$$

$$\Rightarrow (x+2)^2 - y + (y-2)^2 - 4 = 1$$

$$\Rightarrow (x+2)^2 - y + (y-2)^2 = 9$$

* Move all constants to other side of the eqn

$$(x+2)^2 - y + (y-2)^2 = 9$$

* Standard Form

Center: $(\text{-}2, 2)$

Radius: $3$