9.3: Logarithmic Functions

* today we define logarithmic functions (inverse to exponential functions) and discuss their basic properties.

Key to Logarithms: A logarithm is just an exponent.

Goal(s): Evaluate logarithms (Find exponents!) Convert between exponential form & logarithmic form.
Determine the domain of logarithmic functions & graph logarithmic functions. Use to common logarithm & natural logarithm.

Last Time:
- Composition of functions
- Inverse functions
- One-to-one functions
- Solving for inverse functions
- Graphing inverse functions
- The horizontal line test
Recall the graph of the exponential function \( f(x) = b^x \).

**Passes horizontal line test**

- If \( b > 1 \) 
  - \( \text{Passes Horizontal Line Test} \)

In either case, every exponential function is one-to-one. So \( f(x) = b^x \) has an inverse function for any \( b > 0, b \neq 1 \).
* Let's try our switch and solve method for finding the inverse of the exponential function $f(x) = b^x$.

**Step 1:**

$y = b^x$

**Step 2:**

$x = b^y$

**Step 3:**

???

**Step 3 continued:**

New notation $\rightarrow$ **logarithmic notations**

Logarithmic form $\rightarrow$ $y = \log_b(x)$ $\rightarrow$ Gives us a way to solve for $y$...

Equivalent to $\uparrow$

Exponential form $\rightarrow$ $b^y = x$ $\downarrow$

$y$ is the exponent $b$ must be raised by to give $x$.  

* 11/26/16
Step 1: \( y = \log_b(x) \)

\[ \Rightarrow f^{-1}(x) = \log_b(x) \]

(inverse to \( f(x) = b^x \))

* The domain of \( f(x) = b^x \) is \( (-\infty, \infty) \)

* The range of \( f(x) = b^x \) is \( (0, \infty) \)

\[ \Rightarrow (\text{The domain of } f^{-1}(x) = \log_b(x) \text{ is } (0, \infty)) \]

\[ \Rightarrow \text{The range of } f^{-1}(x) = \log_b(x) \text{ is } (-\infty, \infty) \]

The logarithmic function to the base \( b \)

For \( b > 0, b \neq 1 \) the logarithmic function to the base \( b \) is any function of the form:

\[ f(x) = \log_b(x) \]

The domain is \( (0, \infty) \) and the range is \( (-\infty, \infty) \)
Example (Convert: Logarithmic to Exponential Form)

Write each equation in its equivalent exponential form:

a) \( 3 = \log_{7}(x) \)  

\[ y = \log_{b}(x) \iff b^{y} = x \]

\[ y = 3, \ b = 7, \ x = x \]

\[ \rightarrow \ y = 3, \ b = 7, \ x = x \]

b) \( 2 = \log_{b}(26) \)

\[ y = 2, \ b = b, \ x = 26 \]

\[ \rightarrow \ y = 2, \ b = b, \ x = 26 \]

c) \( \log_{y}(26) = y \)

\[ y = y, \ b = y, \ x = 26 \]

\[ \rightarrow \ y = y, \ b = y, \ x = 26 \]
EXAMPLE (Convert Exponential to Logarithmic Form)

Write each equation in its equivalent logarithmic form.

a) \(2^5 = x\)  
    \[
b = x \iff y = \log_b(x)
\]

b) \(b^3 = 27\)  
    \[
b = 2, \; y = 5, \; x = 2^5
\]

b) \(b^3 = 27\)  
    \[
b = 2, \; y = 3, \; x = 2^7
\]

c) \(e^y = 33\)  
    \[
b = e, \; y = y, \; x = 33
\]
Example (Evaluating logarithms) Evaluate:

a) \( \log_{10}(100) \)

b) \( \log_{36}(k) \)

c) \( \log_2\left(\frac{1}{27}\right) \)

d) \( \log_{\frac{1}{64}}\left(\frac{1}{4}\right) \)

* Remember a logarithm is just an exponent

\( a) \quad \log_{10}(100) = 2 \)

To answer this question, we need to determine:

What power must 10 be raised to, to get 100?

\[ 10^2 = 100 \]
b) \( \log_{36} (6) = \frac{1}{2} \)

What power must 36 be raised to, to get 6? 

\[ 36^{\frac{1}{2}} = 6 \quad (\sqrt{36} = 6) \]

c) \( \log_{3} \left( \frac{1}{27} \right) = -3 \)

The exponent 3 must be raised to, to give \( \frac{1}{27} \)? 

\[ -3 = \frac{1}{27} \]

d) \( \log_{\sqrt{64}} \left( \frac{1}{4} \right) = \frac{1}{3} \)

What power must \( \frac{1}{64} \) raise to \( \frac{1}{4} \)? 

\[ \left( \frac{1}{64} \right)^{\frac{1}{3}} = \frac{1}{4} \]
From the basic properties of exponents and the relationship between a function and its inverse...

Basic properties of logarithms

If $b > 0$, $b \neq 1$ and $x > 0$, then

1) $\log_b(b) = 1 \quad \leftarrow \text{since } b^1 = b$

2) $\log_b(1) = 0 \quad \leftarrow \text{since } b^0 = 1$

3) $\log_b(b^x) = x \quad \leftarrow \text{since } \quad \text{// $x$ is the power I must raise } b \text{ to, to get } b^x$

4) $b^{\log_b(x)} = x \quad \leftarrow \text{since } \quad \text{// If I raise } b \text{ to the power which } b \text{ must be raised to, to get } x, \text{ then } I \text{ get } x$. 
Example (using the Basic Properties) Evaluate:

\[ \begin{align*}
\text{a)} & \quad \log_9(9) \\
\text{b)} & \quad \log_8(1) \\
\text{c)} & \quad \log_7(7^8) \\
\text{d)} & \quad 3^{\log_3(17)}
\end{align*} \]

\[ \begin{align*}
\text{a)} & \quad \log_9(9) = 1 \\
\ast & \quad \text{Since } 9^1 = 9 \\
\text{b)} & \quad \log_8(1) = 0 \\
\ast & \quad \text{Since } 8^0 = 1 \\
\text{c)} & \quad \log_7(7^8) = 8 \\
\ast & \quad \text{Since 8 is the power which 7 must be raised to, to give me } 7^8 \\
\text{d)} & \quad 3^{\log_3(17)} = 17 \\
\ast & \quad \text{If I raise 3 to the power 3 must be raised to, to give 17, then I get 17.}
\end{align*} \]
Example: Graph the function \( f(x) = \log_2(x) \)

* We can either plot a few points or ... reflect the graph of \( y = 2^x \) across the line \( y = x \)
Example: Graph the function \( f(x) = \log_{\frac{1}{2}}(x) \)

Again, we'll reflect the graph of \( y = \left(\frac{1}{2}\right)^x \) across the line \( y = x \).

\( y = \left(\frac{1}{2}\right)^x \)
*From these last two graphs, we can determine properties of the graph of \( f(x) = \log_b(x) \) for any \( b > 0, b \neq 1 \).

Characteristics of the Graph of \( f(x) = \log_b(x) \):

1. The domain of \( f \) is all positive real numbers \((0, \infty)\).

2. The range of \( f \) is all real numbers \((-\infty, \infty)\).

3. Since \( \log_b(1) = 0 \), the \( x \)-intercept of the graph of \( f \) is the point \((1, 0)\), where the graph crosses the \( x \)-axis.

4. If \( b > 1 \), then the graph of \( f \) increases from left to right.

5. If \( 0 < b < 1 \), then the graph of \( f \) decreases from left to right.

6. The graph of \( f \) approaches, but never touches, the \( y \)-axis. We say the \( y \)-axis is a vertical asymptote of the graph of \( f \).
Example (Finding the Domain of)
(A logarithmic Function)

Find the domain of the function $f(x) = \log_5(2x+1)$

We know that any input into a logarithm must be a ... Positive Real #

$\Rightarrow$ So to find the domain of $f$ we must solve the inequality:

$2x + 1 > 0$

$\Rightarrow 2x > -1$

$\Rightarrow x > -\frac{1}{2}$

In interval notation the domain of $f$ is $(-\frac{1}{2}, \infty)$
Example (Finding the Domain of) A logarithmic Function

Find the domain of the function \( f(x) = \log_7(3 - 5x) \)

\[
\begin{align*}
\text{Solve: } & \quad 3 - 5x > 0 \\
\Rightarrow & \quad 3 > 5x \\
\Rightarrow & \quad \frac{3}{5} > x
\end{align*}
\]

In interval notation, the domain of \( f \) is

\((-\infty, \frac{3}{5})\)
* The logarithm to the base 10 is called the common logarithm. When writing the Common logarithm we omit the base...

\[ \log(x) = \log_{10}(x) \]

**Example (Earthquake Intensity)** The magnitude, \( R \), on the Richter scale of an earthquake of intensity \( I \) is given by

\[ R = \log \left( \frac{I}{I_0} \right) \]

where \( I_0 \) is the intensity of a (barely felt) zero-level earthquake.
a) What is the magnitude on the Richter Scale of a zero-level earthquake?

b) If an earthquake is 10,000 times more intense than a zero-level earthquake, then what is its magnitude on the Richter Scale?

\[ a) \text{ In this case the magnitude } I \text{ of the earthquake is } I_0 \text{ (since the earthquake is zero-level).} \]

\[ \Rightarrow I = I_0 \]

\[ \text{So } R = \log \left( \frac{I}{I_0} \right) \]

\[ = \log (1) \]

\[ = 0 \]
b) Since the intensity $I$, of the earthquake, is 10,000 as intense as a zero-level earthquake...

$$I = 10,000 I_0$$

So we plug this into the formula for $R$...

$$R = \log \left( \frac{10,000 I_0}{I_0} \right)$$

$$= \log (10,000)$$

$$= 4$$

*Every whole # diffuse of magnitude on the Richter scale corresponds to a 10 time increase in intensity!*
The logarithm to the base \( e \) is called the natural logarithm. When writing the natural logarithm we use the notation

\[
\ln(x) = \log_e(x)
\]

↑

read "el en of x"

Example (Evaluating Natural Logarithms) evaluate:

\[
\begin{align*}
a) & \quad \ln\left(\frac{1}{e}\right) \\
b) & \quad \ln\left(\sqrt{e}\right) \\
c) & \quad e^{\ln(5)} \\
d) & \quad \ln\left(e^{x^2}\right)
\end{align*}
\]

a) \( \ln\left(\frac{1}{e}\right) = \ln\left(e^{-1}\right) = -1 \)

↑

\log base e

\ln(e^{-1})
b) \( \ln(\sqrt{e}) = \ln(e^{\frac{1}{2}}) = \frac{1}{2} \) \\
\Rightarrow \log_{e}(e^{\frac{1}{2}})

c) \( e^{\ln(5)} = 5 \) \\
\Rightarrow e^{\log_{e}(5)} = 5

d) \( \ln(e^{x^2}) = x^2 \) \\
\Rightarrow \log_{e}(e^{x^2})
9.4: Properties of Logarithms

* Now we develop the properties of logarithms further. Again, the main point is:

A LOGARITHM IS AN EXPONENT

so we can adapt the exponent rules to get logarithm rules!

Goal(s): use the product rule, use the quotient rule, use the power rule, expand logarithmic expressions, condense logarithmic expressions, use the change of base formula.
The product rule (For Logarithms)

If \( b, M \& N \) are positive real \#'s, and \( b \neq 1 \), then

\[
\log_b(M \cdot N) = \log_b(M) + \log_b(N)
\]

\[\scriptsize \text{§} \]

IN WORDS

The logarithm of a product, is the sum of the logarithms

\( \text{§} \) This should make sense since the product rule for exponents looks like

\[b^M \cdot b^N = b^{M+N}\]
Example (using the product Rule)

Expand each logarithm using the product rule. Simplify as much as possible:

a) \( \log_7(7 \cdot 11) \)  

\[ \log_7(7) + \log_7(11) \]

\[ = 1 + \log_7(11) \]

\[ \uparrow \]

Fully expanded & simplified

b) \( \log(100x) \)  

\[ \log(100) + \log(x) \]

\[ = 2 + \log(x) \]

\[ \uparrow \]

Fully expanded & simplified
The Quotient Rule (for logarithms)

If \( b, M, N \) are positive real numbers, and \( b \neq 1 \), then

\[
\log_b \left( \frac{M}{N} \right) = \log_b(M) - \log_b(N)
\]

\[\Rightarrow\]

IN WORDS

The logarithm of a quotient is the difference of the logarithms.

\[\Rightarrow\]

This should make sense since the quotient rule for exponents looks like

\[
\frac{b^M}{b^N} = b^{M-N}
\]
Example (Using The Quotient Rule)

Expand each logarithm using the quotient rule, simplify as much as possible:

a) $\log_8 \left( \frac{2^3}{x} \right)$  
b) $\ln \left( \frac{e^5}{11} \right)$

a) $\log_8 \left( \frac{2^3}{x} \right) = \log_8 (2^3) - \log_8 (x)$

Fully expanded and simplified

b) $\ln \left( \frac{e^5}{11} \right) = \ln (e^5) - \ln (11)$

$= 5 - \ln (11)$

Fully expanded & simplified
The Power rule (for logarithms)

If $b$ and $M$ are positive real numbers with $b \neq 1$ and $P$ is any real number, then

$$\log_b(M^P) = P \cdot \log_b(M)$$

**In Words**

The logarithm of a number with an exponent is the product of the exponent and the logarithm of that number.

*This is suggested by the power rule for exponents*

$$(b^M)^P = b^{M \cdot P}$$
Example (using the power rule)

Expand each logarithmic expression using the power rule.

a) $\log_6(8^9)$  
b) $\ln\left(\frac{3}{\sqrt{x}}\right)$  
c) $\log\left((9x)^2\right)$

\[a) \quad \log_6(8^9) = 9 \cdot \log_6(8)\]

\[b) \quad \ln\left(\frac{3}{\sqrt{x}}\right) = \ln\left(x^{\frac{1}{3}}\right) = \frac{1}{3} \cdot \ln(x)\]

\[c) \quad \log\left((9x)^2\right) = 2 \cdot \log(9x)\]
Sometimes it is necessary to combine these rules when expanding a logarithmic expression.

**Example (Expanding Logarithmic Expressions)**

Expand each expression as much as possible:

a) \( \log_b(x^{\frac{4}{3}} \cdot \sqrt{y}) \)

\[
\log_b(x^{\frac{4}{3}} \cdot \sqrt{y}) = \log_b(x^{\frac{4}{3}}) + \log_b(\sqrt{y})
\]

\[
= \log_b(x^{\frac{4}{3}}) + \log_b(y^{\frac{1}{2}})
\]

\[
= \frac{4}{3} \log_b(x) + \frac{1}{2} \log_b(y)
\]

\[
\text{Product rule}
\]

\[
\text{Power rule}
\]

\[
\text{Power rule again}
\]

\[
\text{Fully expanded}
\]

b) \( \log_5\left(\frac{-\sqrt{x}}{25y^2}\right) \)
\[ b) \quad \log_5 \left( \frac{\sqrt{x}}{25y^2} \right) = \log_5 \left( \frac{x^{\frac{1}{2}}}{25y^2} \right) \]

quotient rule

\[ = \log_5 (x^{\frac{1}{2}}) - \log_5 (25y^2) \]

\[ = \frac{1}{2} \log_5 (x) - \left( \log_5 (25) + \log_5 (y^2) \right) \]

\[ = \frac{1}{2} \log_5 (x) - \left( 2 + 2 \log_5 (y) \right) \]

Fundamental relationship & power rule

\[ = \frac{1}{2} \log_5 (x) - 2 \log_5 (y) - 2 \]

distribute the minus sign and rearrange terms.

Fully expanded.