Recall that just like multiplying two rational numbers, the product of two rational expressions is the product of their numerators divided by the product of their denominators.

**Example (Multiplying Rational Expressions)**

Multiply:

\[
\frac{5x+5}{7x-7x^2} \cdot \frac{2x^2+x-3}{4x^2-9}
\]

\[
\frac{5(x+1)}{7(1-x)} \cdot \frac{2x^2-2x+3x-3}{(2x)^2-3^2}
\]

\[
\frac{5(x+1)}{7(1-x)} \cdot \frac{2x(x-1)+3(x-1)}{(2x-3)(2x+3)}
\]
\[ \frac{5(x+1)}{7x(1-x)} \cdot \frac{(2x+3)(x-1)}{(2x-3)(2x+3)} = \frac{5(x+1)(2x+3)(x-1)}{-7x(1-x)(2x-3)(2x+3)} = \frac{5(x+1)(x-1)}{-7x(x-1)(2x-3)} = \frac{-5(x+1)}{7x(2x-3)} \]
Just like dividing two rational numbers, we find the quotient of two rational expressions by inverting the divisor and then multiplying.

\[
\frac{2}{3} \div \frac{9}{5} = \frac{2}{3} \cdot \frac{5}{9} = \frac{10}{12} = \frac{2 \cdot 5}{2 \cdot 3} = \frac{5}{6}
\]

**Example (Dividing Rational Expressions)**

Divide:

\[
(4x^2 - 25) \div \frac{2x + 5}{14}
\]

**Step 1** First, let's factor all numerators and denominators completely:
\[ \frac{x^2 - 25}{1} \div \frac{2x + 5}{14} \]

\[ = \frac{x^2 - 25}{1} \div \frac{2x + 5}{14} \]

\[ = \frac{(2x + 5)(2x - 5)}{1} \div \frac{2x + 5}{2.7} \]

**Step 2:** Invert the divisor and multiply.

\[ = \frac{(2x + 5)(2x - 5)}{1} \cdot \frac{2.7}{2x + 5} \]

\[ = \frac{2.7(2x + 5)(2x - 5)}{2x + 5} \]

**Step 3:** Simplify, if possible.

\[ \frac{7}{1} = 14(2x - 5) \]

\[ = 14(2x - 5) \]
Dividing Rational Expressions

1) Factor all numerators and denominators completely.

2) Invert the divisor and multiply.

3) Simplify the resulting rational expression.

Example (Dividing Rational Expressions)

Divide:

\[
\frac{x^2-5x+6}{2x} ÷ \frac{x^2-5x-6}{x^2-3x}
\]

\[
\frac{x^2-5x+6}{2x} ÷ \frac{x^2-5x-6}{x^2-3x} = \frac{(x-3)(x-2)}{2x} ÷ \frac{(x+1)(x-6)}{x(x-3)}
\]
\[
\frac{(x-3)(x-2)}{2x} \cdot \frac{x(x-3)}{(x+1)(x-6)}
\]

\[
\frac{\sqrt{(x-3)^2(x-2)}}{2x(x+1)(x-6)}
\]

\[
\frac{(x-3)^2(x-2)}{2(x+1)(x-6)}
\]
6.2: Adding & Subtracting Rational Expressions

* When adding or subtracting two fractions we must rewrite the fractions so that they have a common denominator. Usually we look for the smallest common denominator, known as the "least common denominator."

\[
\frac{1}{6} + \frac{1}{9} = \frac{1}{2 \cdot 3} + \frac{1}{3^2}
\]

\[
= \left(\frac{2}{2 \cdot 3}\right) \cdot \frac{1}{2 \cdot 3} + \left(\frac{2}{2}ight) \cdot \frac{1}{3^2}
\]

\[
= \frac{3}{2 \cdot 3^2} + \frac{2}{2 \cdot 3^2}
\]

\[
= \frac{5}{2 \cdot 3^2}
\]

\[
= \frac{5}{18}
\]

* When adding or subtracting rational expressions we must do the same...
Goal(s): Be able to find the least common denominator of two (or more) rational expressions. Be able to add/subtract two (or more) rational expressions.

The least common denominator (or LCD) of two rational expressions is a polynomial consisting of the product of all prime polynomials occurring in each of the denominators, each factor is raised to the greatest power of its occurrence in either denominator. (Let's see an example!)
Example (Finding the LCD)

Find the LCD of \( \frac{3}{10x^2} \) and \( \frac{7}{15x} \)

**Step 1** Factor each denominator completely:

\[
10x^2 = 2 \cdot 5 \cdot x \cdot x \quad 15x = 3 \cdot 5 \cdot x
\]

\[
= 2 \cdot 5 \cdot x^2
\]

**Step 2** Make a list of all factors that appear in each denominator. If a factor is common to both denominators, then list that factor to the highest exponent that it appears in either denominator.

\[
2, 3, 5, \ x^2
\]

**Step 3** The LCD is the product of all factors appearing in the list from **Step 2**;

\[
2 \cdot 3 \cdot 5 \cdot x^2 = 30x^2 \leq \text{LCD}
\]
Example (Finding the LCD)

Find the LCD of \( \frac{x-1}{x+2} \) and \( \frac{3x}{x-2} \)

Step 1: Factor denominators completely:

Already done.

Step 2: Make the list:

\( x+2, x-2 \)

Step 3: The LCD is:

\( (x+2)(x-2) \) \( \leq \) LCD
Example (Finding the LCD)

Find the LCD of \( \frac{9}{7x^2 + 28x} \) and \( \frac{11}{x^2 + 8x + 16} \).

\[
\begin{align*}
7x^2 + 28x &= 7x(x + 4) \\
\text{x}^2 + 8x + 16 &= (x + 4)(x + 4) = (x + 4)^2
\end{align*}
\]

\( \text{LCD} = 7x(x + 4)^2 \)
Finding the LCD of Two Rational Expressions

1) Factor each denominator completely.
   (Write repeating factors with an exponent)

2) Make a list of all factors that appear for each denominator. However, if a factor is common to both denominators only list that factor to the highest exponent that it appears in either denominator.

3) The LCD is the product of all factors from the list in step 2.

Example (Finding the LCD)

Find the LCD of

\[ \frac{2x+1}{x^2-xy+y} \quad \text{and} \quad \frac{3}{2x^3+2x^2-12x} \]
\[ x^2 - 4x + 4 = (x - 2)(x - 2) = (x - 2)^2 \]

\[ 2x^3 + 2x^2 - 12x = 2x(x^2 + x - 6) \]

\[ = 2x(x + 3)(x - 2) \]

\[ \Rightarrow \text{LCD is } 2x(x + 3)(x - 2)^2 \]
Example (Finding the LCD)

Find the LCD of

\[
\frac{5x}{x^2 - 9} \quad \text{and} \quad \frac{8x + 5}{x^2 + 6x + 9} \quad \text{and} \quad \frac{17}{2x^2 + 5x - 3}
\]

* \(x^2 - 9 = (x + 3)(x - 3)\)

* \(x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2\)

* \(2x^2 + 5x - 3 = 2x^2 + 6x - x - 3 = 2x(x + 3) - 1(x + 3) = (2x - 1)(x + 3)\)

Same procedure!
List of factors:

\[(x+3)^2, (x-3), (2x-1)\]

\[\text{LCM: } 11\]

\[\boxed{(2x-1)(x-3)(x+3)^2}\]
Now let's add/subtract rational expressions using the LCD...

**Example (Adding Rational Expressions)**

Add:

\[
\frac{8x}{x^2-16} + \frac{5}{x+y}
\]

**Step 1** Find the LCD of two rational expressions:

\[
x^2-16 = (x+y)(x-y)
\]

\[
x + y
\]

**LCD is** \((x+y)(x-y)\)
Step 2: Multiply the numerator & denominator of both rational expressions by any factor(s) needed to convert the denominator to the LCD:

\[
\frac{8x}{x^2-16} + \frac{5}{x+y} = \frac{8x}{(x+y)(x-y)} + \frac{5}{x+y}
\]

\[
= \frac{8x}{(x+y)(x-y)} + \left(\frac{x-y}{x-y}\right) \cdot \frac{5}{x+y}
\]

\[
= \frac{8x}{(x+y)(x-y)} + \frac{5(x-y)}{(x+y)(x-y)}
\]

\[
= \frac{8x + 5(x-y)}{(x+y)(x-y)}
\]
Step 3: Add the two numerators, placing the resulting expression over the LCD:

\[
\frac{8x + 5(x - y)}{(x+y)(x-y)}
\]

\[
= \frac{8x + 5x - 20}{(x+y)(x-y)}
\]

\[
= \frac{13x - 20}{(x+y)(x-y)}
\]
Step 1) If possible simplify!

Already Simplified
Example (subtracting rational expressions)

Subtract:

\[
\frac{x}{x-3} - \frac{x-1}{x+3}
\]

[Step 1] Find the LCD:

\(x-3\) \(\text{already factored}\)

\(x+3\) \(\text{already factored}\)

\[
\text{LCD} = (x-3)(x+3)
\]
Step 2: Multiply the numerator & denominator of each rational expression by any factor(s) needed to convert the denominator to the LCD:

\[
\frac{X}{X-3} - \frac{X-1}{X+3} = \left(\frac{X+3}{X+3}\right) \left(\frac{X}{X-3}\right) - \left(\frac{X-3}{X-3}\right) \left(\frac{X-1}{X+3}\right)
\]

\[
= \frac{X(X+3)}{(X+3)(X-3)} - \frac{(X-3)(X-1)}{(X+3)(X-3)}
\]

\[
= \frac{X^2 + 3X}{(X+3)(X-3)} - \frac{X^2 - 4X + 3}{(X+3)(X-3)}
\]
Step 3: Subtract the numerators, placing the resulting expression over the LCD:

\[ \frac{x^2 + 3x - (x^2 - 4x + 3)}{(x+3)(x-3)} \]

\[ \frac{x^2 + 3x - x^2 + 4x - 3}{(x+3)(x-3)} \]

\[ \frac{7x - 3}{(x+3)(x-3)} \]
Step 1: It possible, simplify.

Already simplified.
Adding Subtracting Rational Expressions

1) Find the LCD of the rational expressions

2) Rewrite each rational expression as an equivalent expression whose denominator is the LCD. To do this, multiply the numerator and denominator of each rational expression by any factor(s) needed to convert the denominator to the LCD. (Multiplying by 1)

3) Add or subtract numerators, placing the result over the LCD.

4) Simplify if possible
Example (Adding/Subtracting Rational Expressions)

Perform the indicated operations:

a) \[ 7 + \frac{1}{x - 5} \]

b) \[ \frac{x^2 + 2x - 2}{x^2 + 3x - 10} + \frac{5x + 12}{x^2 + 3x - 10} \]

c) \[ \frac{4x + 1}{x^2 + 7x + 12} - \frac{2x + 3}{x^2 + 5x + 4} \]
a) \[ 7 + \frac{1}{x-5} \]

\[ = \frac{7}{1} + \frac{1}{x-5} \]

\[ = \frac{x-5}{x-5} \cdot \frac{7}{1} + \frac{1}{x-5} \]

\[ = \frac{7(x-5)}{x-5} + \frac{1}{x-5} \]

\[ = \frac{7x-35}{x-5} + \frac{1}{x-5} \]

\[ = \frac{7x-35+1}{x-5} \]

\[ = \frac{7x-34}{x-5} \]
\[
\frac{x^2 + 2x - 2}{x^2 + 3x - 10} + \frac{5x + 12}{x^2 + 3x - 10}
\]

\[
\frac{x^2 + 2x - 2 + 5x + 12}{x^2 + 3x - 10}
\]

\[
\frac{x^2 + 7x + 10}{x^2 + 3x - 10}
\]

\[
\frac{(x+2)(x+5)}{(x+5)(x-2)}
\]

\[
\frac{x+2}{x-2}
\]
\[
\frac{4x + 1}{x^2 + 7x + 12} - \frac{2x + 3}{x^2 + 5x + 4} \\
= \frac{4x + 1}{(x+3)(x+4)} - \frac{2x + 3}{(x+1)(x+4)} \\
= \frac{(x+1)(4x-1)}{(x+1)(x+3)(x+4)} - \frac{(x+3)(2x+3)}{(x+1)(x+3)(x+4)} \\
= \frac{(x+1)(4x+1)}{(x+1)(x+3)(x+4)} - \frac{(x+3)(2x+3)}{(x+1)(x+3)(x+4)} \\
= \frac{4x^2 + 5x + 1}{(x+1)(x+3)(x+4)} - \frac{2x^2 + 9x + 9}{(x+1)(x+3)(x+4)} \\
= \frac{4x^2 + 5x + 1 - 2x^2 - 9x - 9}{(x+1)(x+3)(x+4)} \\
= \frac{2x^2 - 4x - 8}{(x+1)(x+3)(x+4)} \\
= \frac{2(x^2 - 2x - 4)}{(x+1)(x+3)(x+4)}
\]
\[ \frac{2x^2 - 4x - 8}{(x+1)(x+3)(x+4)} \]

\[ 2 \left( \frac{x^2 - 2x - 4}{(x+1)(x+3)(x+4)} \right) \]
**Example (Adding Subtracting Rational Expressions)**

Perform the indicated operations:

\[
\frac{3y+2}{y-5} + \frac{4}{3y+4} - \frac{7y^2+24y+28}{3y^2-11y-20}
\]

\[
= \frac{3y+2}{y-5} + \frac{4}{3y+4} - \frac{7y^2+24y+28}{(3y+4)(y-5)}
\]

Continued on next page
\[
\frac{3y+2}{y-5} + \frac{4}{3y+4} - \frac{7y^2 + 24y + 28}{(3y+4)(y-5)}
\]

\[
= \frac{(3y+4)(3y+2)}{(3y+4)(y-5)} + \frac{(y-5)}{(y-5)(3y+4)} - \frac{7y^2 + 24y + 28}{(3y+4)(y-5)}
\]

\[
= \frac{9y^2 + 18y + 8}{(3y+4)(y-5)} + \frac{4y-20}{(y-5)(3y+4)} - \frac{7y^2 + 24y + 28}{(3y+4)(y-5)}
\]

\[
= \frac{9y^2 + 18y + 8 + 4y - 20 - 7y^2 - 24y - 28}{(3y+4)(y-5)}
\]

\[
= \frac{2y - 4}{(3y+4)(y-5)}
\]
\[
\frac{9y^2 + 18y + 8}{(3y+4)(y-5)} + \frac{y^2 + 20}{(3y+4)(y-5)} - \frac{7y^2 - 24y - 28}{(3y+4)(y-5)}
\]

\[
= \frac{2y^2 - 2y - 40}{(3y+4)(y-5)}
\]

\[
= \frac{2(y^2 - y - 20)}{(3y+4)(y-5)}
\]

\[
= \frac{2(y-5)(y+4)}{(3y+4)(y-5)}
\]

\[
= \frac{2(y+4)}{3y+4}
\]