Goal: understand some properties of the graph of $f$.

If $f$ is differentiable at $x_0$ then the graph is like a line through:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$$

When $f'(x_0) = 0$, the formula (1) is not so useful without understanding the $o(h)$ term more.

![Graphs showing different cases](image)

To do this, we use higher derivatives of $f$.

If $f$ is differentiable on $(a, b)$ then $x \mapsto f''(x)$ is a function on $(a, b)$ if $f'$ is continuous, we call $f$ a $C^1$ function. If $f''$ is differentiable we call $f$ twice-differentiable and $f'' = (f''(x))' = f'''(x)$, the third derivative of $f$. Inductively set $f^{(n)}(x) = (f^{(n-1)}(x))' = \lim_{h \to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$ is the $n^{th}$ derivative of $f$. If $f^{(n)}$, $f^{(n+1)}$, ... exist, $f$ is called $n$-times differentiable. If $f^{(n)}$ is continuous, $f$ is called a $C^n$ function.

If $f \in C^1$ then the $f$ is called a $C^0$ or smooth function.

Example: $f(x) = (x^2 \sin(1/x)) \tanh(x)$ is differentiable but not $C^1$.

$x^{1/3}$ is $C^1$ but not $C^{1/2}$ at $x = 0$.

Polynomials are $C^\infty$. 
Polynomial Approximation Let \( f : (a, b) \to \mathbb{R} \) be \( n \)-times differentiable around \( x_0 \in (a, b) \). Then

\[
f(x_0 + h) = f(x_0) + f'(x_0)h + \ldots + \frac{f^{(n)}(x_0)}{n!}h^n + o(h^n)
\]

where \( \frac{o(h^n)}{h^n} \to 0 \) as \( h \to 0 \).

Proof: Use L'Hôpital's Rule.

Prop. If \( f'(x_0) = 0 \) and \( f''(x_0) > 0 \) then \( x_0 \) is a local minimum.

Taylor expansion Let \( f : (a, b) \to \mathbb{R} \) be \( n+1 \)-times differentiable around \( x_0 \in (a, b) \) and \( h > 0 \) s.t. \([x_0-h, x_0+h] \subset (a, b)\). Then

\[
f(x_0 + h) = f(x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}h^n + \frac{f^{(n+1)}(x_0 + \xi h)}{(n+1)!}h^{n+1}
\]

for \( \xi \in (x_0, x_0+h) \). Differentiate to get \( R(t) = -\frac{f^{(n)}(x_0+\xi h)}{n!}(h-t)^n \) and apply L'Hôpital to \( R(t) \) and \( h-n \),

Remark: We often change variables by \( x = x_0 + t \).

Examples: \( f(x) = x^3 \), \( f(x) = \sqrt{x} \), \( f(x) = e^x \).

Exercises: 1. If \( f'''(x_0) = 0 \), \( f''(x_0) = 0 \), \( f'(x_0) = 0 \), \( f(x_0) < 0 \), show that \( x_0 \) is a local maximum.

2. If \( a_0 + a_1 h + \ldots + a_n h^n + o(h^n) = b_0 + b_1 h + \ldots + b_n h^n + o(h^n) \) show \( a_i = b_i \).
3. Find Taylor expansion of \( f \), \( f' \) in terms of coefficients of the expansion for \( f \) and \( g \).

4. How many terms are needed in the Taylor expansion around 0 to approximate \( e \) to 2 decimal places?

5. Show if \( f \) is 2-times differentiable, then

\[
\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).
\]

Newton's method:

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \] should converge to a root of \( f \).

problems: (1) \( f' \) is zero at some \( x_0 \) (2) \( f' \) is bad at a root (\( f(a) = 0 \)) (3) \( f \) has no root (\( f(x) = e^x \)).

A criterion for convergence: Let \( f \) be \( C^1 \) on \( [a, b] \) with a root \( x \in (a, b) \) and \( f' \neq 0 \) on \( [a, b] \). Then \( \exists \) a \( \delta \) such that \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \) converges to \( x \).

proof: Show \( g(x) = x - \frac{f(x)}{f'(x)} \) is a contraction mapping near \( x \), so use Taylor's to get \( |x_{n+1} - x'| \leq C |x_n - x'| \).

\( x^2 + y^2 = 1 \) is locally a graph. Each point has a neighborhood whose

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Implicit function theorem: Let $f: \mathbb{R}^2 \to \mathbb{R}$ be $C^2$ such that $f(x_0, y_0) = 0$, $\frac{df}{dy}(x_0, y_0) \neq 0$. Then $\exists \delta > 0$ and $g: (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ such that $f(x, g(x)) = 0$ and $g(x_0) = y_0$.

Proof: Let $K(x, y) = y - \frac{f(x, y)}{\frac{df}{dy}(x_0, y_0)}$. Show $K$ is a contraction mapping near $(x_0, y_0)$.

Example: 1. $x^2 + y^2 = 1$ has $\frac{df}{dy} = 2y \neq 0$ at any point $(1, 0)$.

2. Differential equation and periodic orbits.

$p(x) - x = 0 \iff$ periodic orbits.
A series is an infinite sum \( a_1 + a_2 + \ldots = \sum_{n=1}^{\infty} a_n \) for some sequence \( (a_n) \). The series converges if \( \sum a_n \) converges and diverges otherwise.

Examples: \( \sum r^n \) for odd \( r \), \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, \( \sum_{n=0}^{\infty} r^n \) diverges.

Methods to test if a series converges or not:

- If \( a_n \geq 0 \) for all \( n \), then \( S_n \) is an increasing sequence. Hence \( \sum a_n \) converges if \( S_n \) is bounded above.

- (Cauchy criterion) \( \sum a_n \) converges if \( \forall \epsilon > 0 \), \( \exists N \) s.t. \( |\frac{a_m}{a_n}| < \epsilon \) for \( m > n \), \( k > 0 \).

- If \( \sum a_n \) converges then \( a_n \to 0 \).

- (Comparison test) If \( \sum a_n \) converges and \( |a_n| \leq b_n \) for \( n \) sufficiently large, then \( \sum b_n \) converges.
  If \( \sum b_n \) diverges and \( a_n \geq b_n \) for \( n \) sufficiently large, then \( \sum a_n \) diverges.

- (Root test) If \( \lim_{n \to \infty} |a_n|^{1/n} < 1 \) then \( \sum a_n \) converges.
  If \( \lim_{n \to \infty} |a_n|^{1/n} > 1 \) then \( \sum a_n \) diverges.

- (Ratio test) If \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} < 1 \) then \( \sum a_n \) converges.
  If \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \) for all \( n \), \( \sum a_n \) diverges.

- (Leibniz test) If \( a_0 > a_1 > a_2 > \ldots \to 0 \) and \( a_n \to 0 \) then \( \sum_{n=0}^{\infty} (-1)^n a_n \) converges.
The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Algebra with series:

1. If $\sum a_n = a$ and $\sum b_n = b$ then $\sum a_n + b_n = a + b$.
2. If $a_n \rightarrow 0$ and $c > 0$ then $\sum c a_n = c \sum a_n$.
3. If $\sum a_n$ converges absolutely and $\sum b_n = b$ then $\sum c a_n = c \sum a_n$ where $c_n = \sum_{k=0}^{n} a_{n-k} b_k$.

Exercises:

1. If $\lim \frac{a_n}{b_n} = c \neq 0$ and $a_n, b_n > 0$ show $\sum a_n$ converges if $\sum b_n$ converges.

2. $\sum \frac{n!}{n^n}, \sum \frac{1}{n^2}, \sum \frac{n}{n^2 - 5n^2}, \sum \frac{1}{n+3^n}, \sum \frac{1}{2^n - n}$

3. If $q_n = \frac{(-1)^n}{n^{1/3}}$. Does it converge absolutely?

Hint: $\sum q_n$. Does it converge conditionally?

How does changing the order of terms affect the sum?

$s = \sum (-1)^n$. Then $\exists s = s + \frac{1}{2} s$ is a rearrangement of $\sum (-1)^n$. But $s + \frac{1}{2} s$ converges to $s$. Therefore, if $\sum a_n$ converges and $\sum b_n$ does not converge then for any $c \in \mathbb{R}$, there exists a rearrangement $\sum a_{\sigma(n)} = c$.

Thus, if $\sum a_n$ converges absolutely then $\sum a_{\sigma(n)} = \sum a_n$ for any rearrangement.
for $f \in C^\infty$ we may write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + \frac{f^{(n)}(x_0)}{(n-1)!} (x-x_0)^{n-1}.$$ 

we'll take $x_0 = 0$ - why? For which $x$ does it make sense to write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n ?$$

Then: if $|f^{(n)}(y)| \leq M$ for all $n \in \mathbb{N}$ and $y \in (-R, R)$ then for each $x \in (-R, R)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$ 

Such functions are called **analytic** about $x_0$.

**Examples:**
1. $e^x = \sum \frac{x^n}{n!}$ a $\mathbb{R}$
2. $\frac{1}{1-x} = \sum x^n \quad \text{on} \quad (-1, 1)$
3. $f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \ 0, & x \leq 0 \end{cases}$ is not analytic about $0$.
4. $f(x) = \int_{0}^{\infty} \frac{e^{-t}}{1+t} dt$ **\text{11}**

**For example, it's known infinitely many primes with $\zeta(1) = \infty$.**

- Abel sums.