Differentiable functions (Rudin ch. 5, Exercise 3)

Let \( f: \mathbb{R} \to \mathbb{R} \) be differentiable at \( x_0 \in \mathbb{R} \), if the limit

\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)
\]

then \( f'(x_0) \in \mathbb{R} \) is called the derivative of \( f \) at \( x_0 \).

Remark: It is equivalent to rewrite \((*)\) as

\[
f(x_0 + h) = f(x_0) + h f'(x_0) + o(h)
\]

where \( o(h) \) stands for some function with \( \frac{o(h)}{h} \to 0 \) as \( h \to 0 \).

We call \( f: (a, b) \to \mathbb{R} \) differentiable on \((a, b)\) if \( f \) is differentiable at each \( x_0 \in (a, b) \).

Examples: \( f(x) = x^2 \) is differentiable on \( \mathbb{R} \), \( f(x) = x^{1/2} \), \( f(x) = 1/|x| \) are not differentiable at \( x = 0 \).

Prop. (1) If \( f \) is differentiable at \( x_0 \) then \( f \) is continuous at \( x_0 \).

(2) If \( f \) and \( g \) are differentiable then so are \( f + g \), \( fg \), \( f g \) with derivative \( f'(x) + g'(x), \ f(x) g'(x) + f'(x) g(x), \ f'(g(x)) g'(x) \) resp. \( \square \)

The function \( f \) has a local maximum at \( x_0 \) if there is some \( \delta > 0 \) s.t. \( f(x) \leq f(x_0) \) for all \( x \in B_\delta(x_0) \). Likewise a local minimum at \( x_0 \) if \( f(x) \geq f(x_0) \) for all \( x \in B_\delta(x_0) \).

Prop. (4) Suppose \( f \) has a local maximum or local minimum at \( x_0 \) and \( f \) is differentiable at \( x_0 \). Then \( f'(x_0) = 0 \).

(2) Suppose \( f'(x_0) > 0 \). Then \( f \) is increasing near \( x_0 \).

(3) Suppose \( f'(x_0) < 0 \) then \( f \) is decreasing near \( x_0 \).
**Rolle's Theorem** Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\) with \( f(a) = f(b) \). Then \( \exists \, c \in (a, b) \) s.t. \( f'(c) = 0 \). □

**Mean Value Theorem** Let \( f : [a, b] \rightarrow \mathbb{R} \) be cont. on \([a, b]\) and diff. on \((a, b)\), then

\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]

for some \( c \in (a, b) \).

**L'Hôpital's Rule:** Let \( f, g : (a, b) \rightarrow \mathbb{R} \) be differentiable with

\[
\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \quad \text{and} \quad \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0.
\]

Then \( \lim_{x \to a} \frac{f(x)}{g(x)} = L \). □

**Exercises:**

1. Show \( \lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h) \).

2. Show product rule.

3. Show \( f(x) = \frac{1}{x} \) is differentiable on \((0, \infty)\).

4. Is \( f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right) & x \neq 0 \\ 0 & x = 0 \end{cases} \) differentiable at \( x = 0 \)?

5. Show \( f(x) = \text{const.} \) is differentiable on \( \mathbb{R} \).

6. Suppose \( f'(x) = 0 \) for all \( x \in \mathbb{R} \), show \( f \) is constant.

7. Suppose \( f \) is differentiable on \( \mathbb{R} \) and \( \lim_{x \to \infty} f'(x) = 0 \).

Show that \( \lim_{x \to \infty} f(x+1) - f(x) = 0 \).

Recall \( x \in \mathbb{R} \) is called a **Liouville number** if for each \( n \in \mathbb{N} \), there exists \( \frac{a_n}{b_n} \in \mathbb{Q} \) (with \( b_n > 1 \)) s.t. \( |x - \frac{a_n}{b_n}| < \frac{1}{b_n^2} \).

**Theorem:** Liouville numbers are transcendental.

**Proof:**

1. Show if \( x \) is a root of \( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Q}[x] \) then \( \exists A > 0 \) s.t. \( |x - \frac{a_n}{b_n}| > \frac{A}{b_n^n} \forall \frac{a_n}{b_n} \in \mathbb{Q} \). □

**Example:** \( \alpha = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \cdots \) is Liouville (and so transcendental).
Is there a continuous function that has a derivative nowhere?

Hard to draw:

As a limit of functions:

How to take limits of continuous functions and have the limit function be cts:

The sequence of functions $f_n : [0, 1] \to \mathbb{R}$ converges pointwise to $f : [0, 1] \to \mathbb{R}$ if for each $x \in [0, 1]$, $f_n(x) \to f(x)$.

Example: $f_n(x) = x^n$ converges pointwise to $f(x) = \begin{cases} 0 & x \in (0, 1) \\ 1 & x = 1 \end{cases}$, not cts!

Set $C([0,1]) = \{ f : [0, 1] \to \mathbb{R} : f \text{ is cts.} \}$ and $\| f - g \| = \sup_{x \in [0,1]} | f(x) - g(x) |$.

The sequence of functions $f_n \in C([0,1])$ converges uniformly to $f : [0, 1] \to \mathbb{R}$ if $\forall \varepsilon > 0, \exists N$ such that $\| f_n - f \| < \varepsilon$ for $n \geq N$.

Thus, $C([0,1])$ with $\| \cdot \|$ is complete (every Cauchy sequence converges) and if $f_n$ converges uniformly to $f$ then $f \in C([0,1])$.

This theorem can be used to show nowhere differentiable functions exist.