# Announcements:
- You can turn HW 3 in next Tuesday if you want more time.
- HW 4 is still due next Thursday.
- If your score on the final is better than your score on the midterm it will replace it.

# Outline for remaining lectures:
- today: continuous functions, Intermediate Value Theorem, Extreme Value Theorem, uniform continuity.
- next week: differentiable functions, Rolle’s Theorem, L’Hospital’s rule.
- Taylor expansions, Implicit function theorem.
- Last week: series, convergence tests, analytic functions.
- Final (comprehensive)

We picture functions \( f: \mathbb{R} \to \mathbb{R} \) by their graphs which are the pairs \( \{(x, f(x)) : x \in \mathbb{R} \} \subset \mathbb{R}^2 \).

\[ f(a) \quad \text{is a graph}, \quad \$ \quad \text{is not a graph} \]

We often consider functions \( f: A \to \mathbb{R} \) which are only defined in some subset \( A \) of \( \mathbb{R} \).

How to define when the graph of \( f \) has no ‘jumps’?

\[ \text{‘jumps'} \]
First we can say when \( f(x) \) gets close to some value \( L \) for \( x \) close enough to some value \( a \):

\[
L \text{ is the limit of } f \text{ as } x \text{ approaches } a \text{ if } \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon.
\]

Definition is: \( \lim_{x \to a} f(x) = L \) or \( f(x) \to L \) (as \( x \to a \)).

Remark: With a limit we are interested in the behaviour of \( f \) for \( x \) near but not necessarily equal to \( a \), this allows us to take limits of functions which may not be defined at \( a \).

Example: \( \lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0 \).

Prop (1) \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = L' \Rightarrow \lim_{x \to a} (f(x)+g(x)) = L + L' \).

(2) \( \lim_{x \to a} f(x) = L \neq 0 \Rightarrow \lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L} \).

(3) \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = L' \Rightarrow \lim_{x \to a} g(f(x)) = L' \).

Proof: \( \lim_{x \to a} f(x) = L \iff \forall \text{ sequence } x_n \to a \text{ with } x_n \neq a \text{ we have } f(x_n) \to L \).

The function \( f: \mathbb{R} \to \mathbb{R} \) is continuous at \( a \) if \( \lim_{x \to a} f(x) = f(a) \). For \( A \subset \mathbb{R} \), \( f: A \to \mathbb{R} \) is continuous on \( A \) if for each \( a \in A \) and \( \varepsilon > 0 \), there is a \( \delta > 0 \) s.t.

\[
|x-a| < \delta \text{ and } x \in A \Rightarrow |f(x) - f(a)| < \varepsilon.
\]

Prop (1) If \( f(x) \) and \( g(x) \) are continuous on \( A \) then \( f(x) + g(x) \) and \( f(x)g(x) \) are also continuous on \( A \).

(2) If \( f(x) \neq 0 \) on \( A \) and \( f(x) \) is cts. on \( A \) then \( \frac{1}{f(x)} \) is cts. on \( A \).

(3) If \( f(x) \) is cts. on \( A \) and \( g(x) \) is cts. on \( f(A) \) then \( g(f(x)) \) is cts. on \( A \).
Examples: \( f(x) = cx \), \( f(x) = x^2 \) are continuous on \( \mathbb{R} \).

Remark: The notation/language hides a lot of information about the function. How does \( \delta \) depend on \( \varepsilon \)? How does \( \delta \) depend on \( a \)?

\[
f \text{ is uniformly continuous on } A \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. for any } x, a \in A \\
| x - a | < \delta \implies | f(x) - f(a) | < \varepsilon.
\]

Prop. Let \( f: A \to \mathbb{R} \) be continuous on \( A \) and \( A \) be a compact set. Then \( f \) is uniformly continuous on \( A \).

Proof (sketch): by continuity we get an open cover \( \{ B_{\delta_i}(a) : a \in A \} \) of \( A \) s.t. \( x \in B_{\delta_i}(a) \Rightarrow f(x) \in B_{\delta_j}(f(a)) \). Take a finite subcover \( B_{\delta_1}(a_1), \ldots, B_{\delta_n}(a_n) \) and set \( \delta = \min \{ \delta_1, \ldots, \delta_n \} \).

- Show that \( x, y \in A \) with \( |x - y| < \delta \) implies \( x, y \in B_{\delta_i}(a_i) \) for some \( i \).
- Use triangle inequality to show \( |f(x) - f(y)| < \varepsilon \). \( \Box \)

Examples: \( f(x) = cx \) is uniformly continuous on \( \mathbb{R} \), \( f(x) = x^2 \) is not uniformly continuous on \( \mathbb{R} \).

Exercises:

1. Show \( \lim_{x \to 1} x^2 = 1 \).

2. If \( f, g: \mathbb{R} \to \mathbb{R} \) are cts and \( f(x) = y(x) \) for all \( y \in \mathbb{R} \), show \( f(x) = g(x) \) for all \( x \in \mathbb{R} \).

3. Show \( f(x) = \frac{1}{x} \) is continuous on \( (0, \infty) \). Is it uniformly continuous?

4. Find a function on \( [0, 1] \) that is continuous and bounded, but not uniformly continuous.

5. Is \( f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases} \) continuous at \( 0 \)?
Intermediate Value Theorem: If \( f : [a,b] \to \mathbb{R} \) is cts. with \( f(a) < 0 < f(b) \) then
\[
\exists c \in (a,b) \text{ s.t. } f(c) = 0.
\]

**Proof (sketch):** Set \( U = \{ x \in [a,b] : f(x) < 0 \text{ for } x \in [a,b] \} \) and take \( c = \sup U \). Show \( f(c) = 0 \) using the following Lemma:

**Lemma:** If \( f \) is continuous at \( x_0 \) and \( f(x_0) > 0 \) then \( \exists \delta > 0 \text{ s.t. } f(B_{\delta}(x_0)) > 0 \).

Extreme Value Theorem: If \( f : [a,b] \to \mathbb{R} \) is cts. then \( \exists c \in [a,b] \) s.t.
\[
f(c) \geq f(x) \text{ for all } x \in [a,b].
\]

**Proof (sketch):** Set \( U = \{ x \in [a,b] : f \text{ is bounded on } [a,u] \} \), show \( \sup U = b \) using the following Lemma:

**Lemma:** If \( f \) is cts. at \( x_0 \) then \( \exists \delta > 0 \text{ s.t. } f \text{ is bounded on } B_{\delta}(x_0) \).

This shows \( f \) is bounded, it remains to show the maximum = \( f(c) \) for some \( c \).

Set \( y = \sup \{ f(x) : x \in [a,b] \} \) has \( y \geq f(x) \text{ for all } x \in [a,b] \).

Then \( g(x) = \frac{1}{y - f(x)} \) is cts. on \([a,b]\) and is bounded, use this to contradict that \( y \) is the least upper bound of \( \{ f(x) : x \in [a,b] \} \).

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**Exercises:**

1. If \( f : [a,b] \to \mathbb{R} \) has \( f(a) > C > f(b) \) then \( f(c) = C \) for some \( c \in (a,b) \).
2. Find counterexamples to the IVT/EVT if \( f \) is not cts.
3. Find a counterexample to the EVT if \( f : (a,b) \to \mathbb{R} \).
4. Show the IVT does not hold for \( f(x) = \frac{1}{x^2 + 1} : \mathbb{Q} \to \mathbb{Q} \) (only consider rational \#s).
5. \( f : [a,b] \to \mathbb{R} \text{ cts. } \Rightarrow \exists c \in [a,b] \text{ s.t. } f(c) \leq f(x) \forall x \in [a,b] \).
6. \( f(a) < g(a) \) and \( f(b) > g(b) \) and \( f, g : [a,b] \to \mathbb{R} \text{ cts. } \Rightarrow f(c) = g(c) \text{ for some } c \in (a,b) \).
7. \( f : [a,b] \to [0,1] \text{ cts. } \Rightarrow f(c) = c \text{ for some } c \in [0,1] \).
8. How many continuous functions are there with \( f(x)^2 = x^2 \) ?
9. If \( f : \mathbb{R} \to \mathbb{R} \) is continuous and \( f(x) \geq 0 \forall x \), what can you say about \( f \)?