HW5. Due Thursday 7/27

1. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is differentiable at \( x_0 \). Show that \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x-x_0} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \).

2. Suppose \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( |f(x) - f(y)| \leq (x - y)^2 \) for all \( x, y \in \mathbb{R} \). Show that \( f \) is constant

3. Suppose \( f, g \) are differentiable on \( \mathbb{R} \) with \( f'(x) = g'(x) \) for all \( x \in \mathbb{R} \). Show that \( f(x) = g(x) + c \) for some constant \( c \in \mathbb{R} \).

4. If \( C_i \in \mathbb{R} \) are some numbers with \( C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0 \) show there exists some \( x \in (0, 1) \) s.t. \( C_0 + C_1x + \ldots + C_nx^n = 0 \).

5. Show that \( f_n(x) = x^n : [0, 1] \to \mathbb{R} \) converge pointwise to a discontinuous function on [0, 1].

6. Let \( f \) be \( n \) times differentiable around \( x_0 \) with \( n \) odd. Suppose \( f^{(1)}(x_0) = f^{(2)}(x_0) = \ldots = f^{(n-1)}(x_0) = 0 \) and \( f^{(n)}(x_0) > 0 \). Show that \( f \) is increasing on some neighborhood of \( x_0 \).

7. Suppose \( f \) is differentiable on \( (a, b) \) and \( f'(x) \neq 0 \) on \( (a, b) \). Show that either \( f'(x) > 0 \) for all \( x \in (a, b) \) or \( f'(x) < 0 \) for all \( x \in (a, b) \). (note, we are not assuming \( f' \) is continuous)

8. Let \( f : (a, b) \to \mathbb{R} \) have \( f'(x) > 0 \) for all \( x \in (a, b) \).
   (a) Show that \( f \) is invertible on \( (a, b) \).
   (b) Show that the inverse of \( f \), \( f^{-1} \) is differentiable on \( f((a, b)) \)

9. Let \( f \) be \( C^n \) on the interval \( (a, b) \) and \( f^{(n)}(x) \neq 0 \) for all \( x \in (a, b) \). Show \( f \) has at most \( n \) zeroes in \( (a, b) \).
   (as a corollary we can deduce that a polynomial of degree \( n \) has at most \( n \) roots)

10. Let \( f \) be \( C^2 \) on \( (a, b) \) with \( f'(x) \neq 0 \) for all \( x \in (a, b) \) and suppose \( f \) has a root \( r \in (a, b) \) (so \( f(r) = 0 \)).
    Take \( x_0 \in (a, b) \) and define the sequence \( x_n \) recursively by \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \). We showed in class that this sequence goes to \( r \), so remains in \( [r - \delta, r + \delta] \subset (a, b) \) for some \( \delta > 0 \).
    By evaluating the first degree Taylor expansion around \( x_n \):
    \[ f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(q)}{2}(x - x_n)^2 \]
    at \( x = r \), deduce that \( |x_{n+1} - r| \leq C(x_n - r)^2 \) for some constant \( C \).