Nash Bargaining versus Market Outcomes*

Nirvikar Singh**

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Abstract

This paper compares the NBS and market outcomes in a simple n-person economy. It shows how the two outcomes differ with respect to responsiveness to differences in risk aversion, endowments, and market positions.

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**Department of Economics, Social Sciences 1, University of California, Santa Cruz, CA 95064. Phone: (831) 459-4093, Fax: (831) 459-5900, email: boxjenk@cats.ucsc.edu

A Note on Bargaining and Market Outcomes

Nirvikar Singh
Dept. of Economics, UC Santa Cruz

Introduction

The Nash Bargaining Solution (NBS) (Nash, 1950) is an appealing cooperative solution concept for bargaining games, in part because it approximates the equilibrium of a well-defined noncooperative bargaining procedure (Binmore, 1980; Rubinstein, 1982; Binmore, Rubinstein and Wolinsky, 1986; Krishna and Serrano, 1996). This approximation result holds where utility is transferable, so a fixed surplus is being divided. The popularity of the NBS is possibly also driven by its simplicity: it involves maximizing a simple objective function, and is easily calculated. The NBS is a component of theories ranging from those explaining the nature of the firm (Grossman and Hart, 1986) to those examining trade wars and trade agreements (Grossman and Helpman, 1995).

The n-person version of the NBS is a natural extension of Nash's original formulation (Harsanyi, 1977, Ch. 10), derived either through consistency requirements with the set of bilateral bargaining subgames, or through direct extensions of Nash's axioms. In the n-player case, it is also plausible to think of resource allocation being governed by market outcomes with price-taking behavior, rather than by bargaining. Such behavior is typically motivated by individuals being small relative to the market (n is large), and competitive equilibrium with price-taking is, of course, a major aspect of the economic theory of resource allocation.

This paper compares the NBS and market outcomes in a simple n-person game. The NBS may be thought of as approximating the n-person noncooperative bargaining game of Krishna and Serrano (1996), which generalizes the Rubinstein two-person bargaining game in an appealing manner. The market outcome, or price-taking equilibrium, as usual, is taken as an approximation to the outcome of some institutional set-up involving bids and asks, posted prices, etc. The purpose of the analysis is to gain some insight into the workings of these two equilibrium concepts. The analysis is modest in scope, and does not attempt to model pairwise matching mechanisms, as has been done in several papers exploring the relationship between bargaining and markets (Gale, 1987; Binmore and Herrero, 1988; Rubinstein and Wolinsky, 1990).

Model

There are n individuals, indexed by subscript i, i = 1, …, n. Their utilities are u_i, and their utilities in the case of no agreement are d_i. In order to compare the bargaining and market outcomes, we take the simplest possible case of trade. There are two goods, *w* and *y*, and individual i's utility function is given by

(1) $$u_i(w_i, y_i) = v_i(w_i) + y_i\,.$$

Hence the utility function is quasilinear, and utility is fully transferable. We assume that v_i is twice-differentiable and strictly concave, and that $v_i(0) = 0$ for all *i*. We denote the initial allocations (endowments) and utilities by w_i^O, y_i^O, u_i^O; the NBS allocations and utilities by w_i^B, y_i^B, u_i^B; and the competitive market outcomes and utilities by w_i^M, y_i^M, u_i^M. Hence, in particular, the disagreement payoffs, d_i, are exogenously given by u_i^O, rather than being endogenously determined in any way (as, for example, in Nash's 1953 theory of rational threats).

Let the total surplus in the case of agreement be denoted by *S*. This quantity is a constant, due to the quasilinear utility functions. Since the bargaining solution is Pareto efficient,

(2) $$S = \sum_1^n (v_i(w_i^B) + y_i^B) - \sum_1^n (v_i(w_i^O) + y_i^O)\,.$$

Since the competitive market outcome is also Pareto efficient, the total surplus (or gains from trade) is the same in that case. Hence the superscript 'B' in the last equation can equivalently be replaced by 'M'. Note that, since there is only trade and no production, the aggregate quantity of *y* is fixed, and (2) can be simplified to

(2)′ $$S = \sum_1^n v_i(w_i^B) - \sum_1^n v_i(w_i^O)\,.$$

Furthermore, the quasilinear utility functions imply that, at any Pareto efficient allocation, $v_i'(w_i) = v_j'(w_j)$ for all *i*, *j*, *i* … *j*. Hence, from strict concavity of the v_i's, $w_i^B = w_i^M$ for all *i*.[1] Only the allocation of the numeraire *y* will differ between the NBS and the market outcome.

Nash Bargaining Outcome

The NBS is the solution to maximizing the product $\prod_1^n (u_i - u_i^O)$, subject to individual rationality and feasibility. The NBS is easily shown to satisfy

(3) $$u_i^B = u_i^O + \tfrac{1}{n} S\,.$$

[1] Suppose not, in which case $w_i^B \neq w_i^M$ for some *i*. Then the marginal utilities differ for all individuals across the bargaining and market solutions, since they are equal across individuals within each case. But then the total amount of *w* cannot be equal across the two allocations.

Hence, the NBS involves an equal sharing of the surplus that results from the Pareto efficient reallocation of the two goods, *w* and *y*. This is, of course, a well-known property.

Competitive Market Outcome

Let *y* be the numeraire good in the competitive market case, and let the price of *w* be p_w. Then in the competitive equilibrium, each individual maximizes u_i subject to the budget constraint

(4) $$p_w(w_i - w_i^O) + (y_i - y_i^O) = 0 .$$

In equilibrium, therefore, for each individual i,

(5) $$p_w = v_i'(w_i^M),$$

and the net transfer of the numeraire is[2]

(6) $$t_i^M = (y_i^M - y_i^O) = -v_i'(w_i^M)(w_i^M - w_i^O) .$$

The utility of individual i in the competitive market equilibrium is, therefore,

(7) $$u_i^M = v_i(w_i^M) - v_i'(w_i^M)(w_i^M - w_i^O) + y_i^O ,$$

and the utility gain of individual i is

(8) $$u_i^M - u_i^O = v_i(w_i^M) - v_i(w_i^O) - v_i'(w_i^M)(w_i^M - w_i^O) .$$

It is easy to check that this expression is positive, whatever the direction of the trade, when v_i is strictly concave. It is clear that equations (3) and (8) are very different in structure, and our goal is to compare these two outcomes.

Results

We first consider the effect of endowments on the gains from trade. While equation (3) indicates that the utility gain in the NBS is the same for each individual, and independent of the nature of preferences, this is not the case for the competitive market outcome. In fact, individual endowments systematically affect the market outcome. In particular, we have Proposition 1.

Proposition 1

If individuals have identical utility functions, then those who make the smallest (largest) trades gain the least (most) in the competitive market outcome.

[2] We assume individual endowments are sufficient to ensure that no individual is constrained to a corner outcome.

Proof

If utility functions are identical, then, from (5), in the competitive market equilibrium, w_i^M is identical for all individuals. Hence (8) becomes

$$u_i^M - u_i^O = v(w^M) - v(w_i^O) - v'(w^M)(w^M - w_i^O), \tag{9}$$

where we drop the subscript on the w^M.

Let $f(w_i^O) = v(w_i^O) - v'(w^M)w_i^O$. Hence, (9) becomes

$$u_i^M - u_i^O = v(w^M) - v'(w^M)w^M - f(w_i^O) \tag{10}$$

We have $f'(w_i^O) = v'(w_i^O) - v'(w^M)$ and $f''(w_i^O) = v''(w_i^O)$. Since $v''(w_i^O) < 0$, f is strictly concave with a maximum at w^M. Hence, from (10), $u_i^M - u_i^O$ is strictly convex in w_i^O, with a minimum at w^M. In other words, the utility gain is smallest (largest) for those whose trades are smallest (largest). ∎

Corollary 1

In the case of Proposition 1, individuals who trade more than average in the competitive market equilibrium are better off than in the NBS.

Proof

Since the total utility gain from reallocation of the goods is the same in the two cases, the result follows immediately. ∎

It might seem that this result is a straightforward consequence of the symmetry required in the Nash axioms: utility gains are equal in the NBS, whereas in the competitive equilibrium, it is marginal utility that matters. However, if we think of the NBS as approximating a noncooperative bargaining procedure, then the equal gains can be thought of as a consequence of equality in the bargaining procedure. It then seems more noteworthy that the competitive market outcome does not lead to this kind of equal gain (being driven by the different logic of marginal utility).

Corollary 1 compares the NBS and the competitive outcome when preferences are identical but endowments are different. One may also ask what happens when endowments are identical but preferences are different. In the case of quasilinear preferences, the natural dimension of variation of preferences is in risk aversion: for example, the coefficient of absolute risk aversion is given by $A_i = -v_i''/v_i'$. Treating A_i as a parameter that captures variation in preferences, and assuming identical endowments, equation (8) becomes

(11) $$u_i^M - u_i^O = v(w_i^M, A_i) - v(w^O, A_i) - v'(w_i^M, A_i)(w_i^M - w^O)$$

Furthermore, from (5), $p_w = \partial v(w_i^M, A_i)/\partial w$, so we may write $w_i^M = w^M(p_w, A_i)$. We may substitute from (5), so that (11) becomes

(12) $$u_i^M - u_i^O = v(w_i^M, A_i) - v(w^O, A_i) - p_w(w_i^M - w^O)$$

Differentiating (12) with respect to A_i, we have

(13) $$\frac{\partial}{\partial A_i}(u_i^M - u_i^O) = \frac{\partial}{\partial w}v(w_i^M, A_i)\frac{\partial w^M}{\partial A_i} + \frac{\partial}{\partial A_i}v(w_i^M, A_i) - \frac{\partial}{\partial A_i}v(w^O, A_i) - p_w\frac{\partial w^M}{\partial A_i}$$

Again using (5), the first and last terms on the RHS of (13) cancel out, so it reduces to

(14) $$\frac{\partial}{\partial A_i}(u_i^M - u_i^O) = \frac{\partial}{\partial A_i}v(w_i^M, A_i) - \frac{\partial}{\partial A_i}v(w^O, A_i).$$

We may now derive the following result.

Proposition 2

If individuals in the economy have identical endowments but differing levels of *constant* absolute risk aversion, then the following holds:

(i) If relative risk aversions are greater than 1, then the gains from trade are smaller for more risk averse buyers of w and greater for more risk averse sellers.

(ii) If relative risk aversions are less than 1, then the gains from trade are greater for more risk averse buyers of w and smaller for more risk averse sellers.

Proof

In the case of constant absolute risk aversion, the (canonical) utility function is given by $v(w, A) = -e^{-Aw}$.

We first establish a relationship between risk aversion and who is a net buyer or seller of w. From (5), $w(p, A) = \frac{1}{A}\ln(\frac{A}{p})$.

Hence, $w_A(p, A) = \frac{1}{A^2}\left[1 - \ln(\frac{A}{p})\right] = \frac{1}{A^2}\left[1 - Aw\right]$.

Since $R = Aw$ is the coefficient of relative risk aversion, we have that

(15) $w_A(p, A) > (<)\ 0$ as $R < (>)\ 1$.

In words, the optimized level of w increases (decreases) in A for R <(>) 1. Since all individuals have the same initial endowment, it follows that there is a cutoff value A^* for the risk aversion parameter such that, if R < (>) 1, then individuals are net buyers (sellers) if and only if $A > A^*$.

We next examine how the gains from trade vary with the level of risk aversion. Using (14) and the CARA functional form, we have

$$\frac{\partial}{\partial A_i}(u_i^M - u_i^O) = w_i^M e^{-A_i w_i^M} - w^O e^{-A_i w^O} \quad (16)$$

Furthermore, $\frac{\partial}{\partial w} we^{-Aw} = e^{-Aw}(1 - Aw) = e^{-Aw}(1 - R)$, so that the sign of this latter derivative depends on whether R <(>) 1.

Hence, from (15) and (16), and the last equalities, we have the following four cases:

	$R > 1$	$R < 1$
$w_i^M > w^O$ (buyer)	$\frac{\partial}{\partial A_i}(u_i^M - u_i^O) < 0$	$\frac{\partial}{\partial A_i}(u_i^M - u_i^O) > 0$
$w_i^M < w^O$ (seller)	$\frac{\partial}{\partial A_i}(u_i^M - u_i^O) > 0$	$\frac{\partial}{\partial A_i}(u_i^M - u_i^O) < 0$

This proves the result as stated.

Since, irrespective of the magnitude of R relative to 1, it is those with larger equilibrium trades who have higher gains from trade,[3] we also have the following consequence of the equality of gains in the NBS.

Corollary 2

In the economy of Proposition 2, those individuals with trades above a certain level (i.e., those with more extreme risk aversion parameters) are better off with the competitive market outcome than in the NBS.

It is possible to conceive of examples where the utility gain of individuals with less extreme risk aversion is higher in the competitive equilibrium. The more general point, therefore, is that the utility gains in the market outcome are sensitive to these risk attitudes, while they are irrelevant for the NBS.

[3] For example, if $R < 1$, w is increasing in A, so that larger buyers are more risk averse, while larger sellers are less risk averse. The statement relating gains from trade to size of trade then follows from the second column of the table.

Another significant way that the NBS and the competitive market outcome differ is in the effects of relative scarcity of w_i. If the total endowment $\sum_1^n w_i^O$ decreases, then p_w, which is the (shadow) price of this good[4], must increase, reflecting the increased relative scarcity of w. From (2)′, we see that S, the total gains from trade, may increase or decrease in general, though each part of the right-hand side will go down. The precise effect will depend on how the endowment reduction is distributed. However, it remains true, from equation (3), that the gains from Nash bargaining are equally distributed.

The situation is quite different in the case of the competitive equilibrium. In that case, the gains from trade depend very heavily on being a net buyer or seller of w_i. We have Proposition 3.

Proposition 3

A marginal decrease in the total endowment of w increases the utility gain of sellers whose endowment does not decrease and decreases the utility gain of buyers of the good whose endowment does not increase.

Proof

Let $W^O \equiv \sum_1^n w_i^O$ be the total endowment. We differentiate equation (8) with respect to this variable, obtaining

$$(17)\qquad \frac{\partial}{\partial W^O}(u_i^M - u_i^O) = v_i'(w_i^M)\frac{\partial w_i^M}{\partial W^O} - v_i'(w_i^O)\frac{\partial w_i^O}{\partial W^O} - v_i''(w_i^M)(w_i^M - w_i^O)\frac{\partial w_i^M}{\partial W^O} - v_i'(w_i^M)\left(\frac{\partial w_i^M}{\partial W^O} - \frac{\partial w_i^O}{\partial W^O}\right)$$

Canceling out and gathering terms, we have

$$(18)\qquad -v_i''(w_i^M)(w_i^M - w_i^O)\frac{\partial w_i^M}{\partial W^O} + (v_i'(w_i^M) - v_i'(w_i^O))\frac{\partial w_i^O}{\partial W^O}$$

Now,

$$\frac{\partial w_i^M}{\partial W^O} = \frac{\partial w_i^M}{\partial p_w}\frac{\partial p_w}{\partial W^O}.$$

From equation (5),

[4] The value of p_w is determined by the set of n equations (5) plus the resource constraint $\sum_1^n w_i = \sum_1^n w_i^O$

$$1 = v_i''(w_i^M)\frac{\partial w_i^M}{\partial p_w}.$$

Hence, (18) becomes

$$(19) \qquad -(w_i^M - w_i^O)\frac{\partial p_w}{\partial W^O} + (v_i'(w_i^M) - v_i'(w_i^O))\frac{\partial w_i^O}{\partial W^O}.$$

The simplest case to consider is where the individual endowment is unchanged. Then the second part of (19) vanishes, and, since $\frac{\partial p_w}{\partial W^O} < 0$ because of the increased relative scarcity of the good, the utility gain decreases for buyers of *w* and increases for sellers whose endowments are unchanged, as a result of a decrease in the total endowment. The remainder of the proposition is demonstrated by noting that the term in the second set of parentheses is positive for sellers and negative for buyers. The appropriate sign of the last derivative then reinforces the effect of the first term. ∎

The interest of the above result lies in the comparison of the NBS and market outcome. The following corollary is a straightforward consequence of Proposition 3.

Corollary 3

If a marginal reduction in the total endowment of *w* is made entirely at the expense of buyers of the good in a competitive equilibrium, then the gains from trade of all buyers (sellers) are lowered (raised). In contrast, in the NBS, such a reduction in the endowment has the same effect on the utility gain of all individuals.

Yet another interesting comparison between the NBS and market outcome is in their different responses to transfers of endowment from one individual to another. In this simple model, an increase in an individual's endowment at the expense of another can only increase that person's final utility. However, the effects on utility *gains* are subtler. This is described in the following result.

Proposition 4

If individual preferences are identical, and individual i receives a small transfer of endowment of *w* from individual j, then

(a) In the competitive outcome, i's utility *gain* increases if he is a net seller of *w* and decreases if he is a net buyer.

(b) In the NBS, i's utility *gain* increases if his initial endowment exceeds that of j, and decreases if it is less than that of j.

Proof

From (5), with identical preferences and strict concavity, competitive equilibrium allocations of *w* are all equal. This is also true for the NBS. If the total endowment of *w* is unchanged, then final allocations of *w* are unchanged for both the market outcome and the NBS. Given these facts, and the nature of the transfer, the effects on utility gains are:

$$\frac{\partial}{\partial w_i^o}(u_i^M - u_i^O) = -v'(w_i^O) + p_w,$$

by differentiating (8) and using (5),

and

$$\frac{\partial}{\partial w_i^o}(u_i^B - u_i^O) = -\tfrac{1}{n}[v'(w_i^O) - v'(w_j^O)],$$

by differentiating (3), and using (2)′ and the fact that $\delta w_i^O = -\delta w_j^O$.

The results now follow from the strict concavity of *v*. ∎

Note that the result is in keeping with the different underlying forces in the two outcomes: for the market outcome, i's position in the market (buyer or seller) is what matters, not his position relative to individual j. On the other hand, only that relative initial position matters for the NBS, while the direction of trade is irrelevant.

Conclusion

The results presented in this paper are not surprising, but they are appealing. While it is well known that the competitive market outcome and the NBS outcome are very different, not much has been written on understanding the nature of these differences. This paper provides some insights in this direction. It may be observed that much of the "action" in the propositions comes from variations in the market outcome. The NBS here involves just an equal split of the total gains from trade. One can generalize the NBS to allow for differences in bargaining power (e.g., Harsanyi and Selten, 1972), so that the generalized NBS maximizes $\prod_1^n (u_i - u_i^O)^{\alpha_i}$. If the α_i's, which measure relative bargaining power, are normalized to sum to one, then individual i's share of a given surplus is equal to α_i. Given arbitrary exogenous α_i's, the results of this paper can be suitably modified to this case. An interesting question for future research is whether the α_i's can be related to preferences or endowments in any systematic way. Binmore, Rubinstein and Wolinsky (1986) have related the bargaining strength parameters to time preferences or probability estimates of bargaining breakdown when the generalized NBS is viewed as an approximation to a Rubinstein (1982) alternating offers noncooperative bargaining model. However, that approach does not address the kinds of variations considered in this paper.

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