QUATERNIONS

BRETT HOUTZ

Quaternions are numbers that are often thought of as an extension of complex numbers. In the latter half of the 19th century, quaternions helped give rise to the modern concept of the vector. Their most significant application is in the computation and representation of three-dimensional rotations.

1. History

1.1. Hamilton. Quaternions were discovered by Sir William Rowan Hamilton in 1843. Hamilton was born in 1805 in Dublin, Ireland, where he would spend much of his life. He was something of a child prodigy; he placed first in the entrance exam to Trinity College, and was named Andrews’ Professor of Astronomy at the University of Dublin and Royal Astronomer of Ireland as an undergraduate. Hamilton went on to lead a successful career in mathematics and physics, publishing influential works on the theory of optics. Hamilton was knighted in 1835. [1]

1.2. Algebra of Triples. Around the turn of the 19th century, several mathematicians independently published works demonstrating interpretations of complex numbers as points on a plane. Inspired by these results, Hamilton did work on the interpretation of complex numbers as pairs of real numbers in the 1830s. He found it fascinating that, according to this interpretation, points in the Euclidean plane could be subject to all the same operations as real numbers — complex numbers are algebraic pairs. [1]

It is important to note that at this point in history, the contemporary theory of vector spaces did not exist at all. The commutative, associative, and distributive laws of algebraic operations were also rarely acknowledged at this time. They were simply taken for granted. Hamilton, however, demonstrated an awareness of these laws and their importance in his work that was ahead of his time. [1]

Hamilton sought to discover a “Theory of Triplets”: numbers with three parts that behave similarly to complex numbers. He essentially desired what today would be called a three-dimensional field1 over the real numbers. He tried fruitlessly to create such numbers with a second \( \sqrt{-1} \), in addition to \( i \). No definition of the product of \( i \) and \( j \) would yield a well-behaved algebra. In October 1843, Hamilton discovered that defining the product of \( i \) and \( j \) as one more imaginary part \( k \) permits well-defined arithmetic operations. [2] In a letter to his son, Hamilton recounts the moment he finally came upon this discovery while on a walk with his wife:

“An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should ever be allowed to live

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1 A field may essentially be understood as a set for which addition, subtraction, multiplication, and division work just as they do on real numbers.
long enough distinctly to communicate the discovery. I pulled out on the spot a pocket-book, which still exists, and made an entry there and then. Nor could I resist the impulse — unphilosophical as it may have been — to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols

\[ i, j, k; i^2 = j^2 = k^2 = ijk = -1 \]

Hamilton’s etching into Dublin’s Broom Bridge has since eroded, but a plaque with the formula was installed to commemorate Hamilton. In his earliest communications about this discovery, Hamilton had already named these numbers of four parts (three imaginary and one real) quaternions. 

Quaternions have all the properties of a field — except the commutativity of multiplication. Though non-commutative algebras are commonplace in contemporary mathematics, the labelling of a non-commutative operation as “multiplication” was novel in Hamilton’s time. Still, quaternion algebra works quite well, and Hamilton considered it a major discovery. He would spend the rest of his life researching quaternions. In the course of his work on quaternions, Hamilton gained a fervent disciple, Peter Tait, who would be the main source of research and publications on quaternions after Hamilton’s death in 1865.

1.3. Birth of the Vector. Though Hamilton never succeeded in finding a pure algebra of triples, he was eager to interpret the imaginary part (the coefficients of \( i, j, \) and \( k \)) as representing points in three-dimensional space. Hamilton’s name for the imaginary part was the vector part, and his name for the real part the scalar part. Until then, the word “vector” had only been used in the compound term “radius vector.” The usage of vector to describe an arbitrary directed magnitude in space started in the context of quaternions.

In the context of three-dimensional Euclidean space, there is a convention of denoting the standard coordinate unit vectors \( i, j, k \). This notation originates in quaternions. Hamilton and many of his contemporaries did not think it was natural to consider tuples of numbers as something on which computations should be done — their existence should be justified by natural algebraic laws. With the cross and dot products built into the quaternion product (as will be shown later), pure imaginary quaternions seemed a natural fit to represent points in space. Thus in the decades following the invention of quaternions, pure imaginary quaternions were the de facto representation of vectors in the English-speaking world.

The use of quaternions in this way is illustrated in Hamilton’s proof of the Cayley-Hamilton Theorem. The theorem states that the characteristic polynomial of a linear transformation, when applied to that transformation, yields the zero function. Hamilton earned his place in the name by proving the theorem in the case of linear transformations on \( \mathbb{R}^4 \). However, since the formal theory of vector spaces had not yet been developed, Hamilton actually showed the result on the space of quaternions with both real and imaginary parts. He did not utilize any of the properties that make quaternions unique from \( \mathbb{R}^4 \) - quaternions were simply the most natural way of expressing quadruples of real numbers.

Some claim that Hamilton coined “vector,” but querying Google Books’ Ngram Viewer clearly shows that this isn’t true. The results do indicate that he was the first to use the word in isolation from “radius”.

While the usage of vector originates from Hamilton, other early conceptualizations of vectors were emerging independently in other parts of the Europe during the mid-19th century.
James Clerk Maxwell used the quaternion product — or rather, its scalar and vector parts — in his work on the theory of electromagnetism. In an 1871 paper on the classification of quantities on physics, he commended the invention of quaternions, with reservations:

“A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into Scalar quantities, which are completely represented by one numerical quantity, and Vectors, which require three numerical quantities to define them.

The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared, for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.”

The “ideas of this calculus” refer to the concept of a vector as a quantity; its “operations and symbols” refer to the idea that vectors are square roots of $-1$, and the definition of the quaternion product having both a scalar part and a vector part. The scalar and vector products, which would later be known as the dot and cross products, were of great use, but a large part of Maxwell’s paper discussed the importance of distinguishing between scalars and vectors. Abstract quantities with both scalar and vector parts are of little use to practical physics. [5]

As the 19th century progressed, the use of vectors would rise in popularity but the use of quaternions, the quantities from which they originated, would fall. By the turn of the 20th century, vectors had become a concept widely regarded to be completely separate from quaternions. [1]

2. Definition and Properties

2.1. Contemporary Definition. Quaternions are quadruples of real numbers $(a, b, c, d)$, written as $a + bi + cj + dk$, with addition

\[ q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k \]

and multiplication

\[
q_1q_2 = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \\
+ (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\
+ (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j \\
+ (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k
\]

The set of quaternions is denoted $\mathbb{H}$ (after Hamilton).

2.2. Original Definition. Quaternions are complexes of real numbers and distinct imaginary quantities $i, j$, and $k$ with the property:

\[ i^2 = j^2 = k^2 = ijk = -1 \]

Quaternions may be added and multiplied as if they are expressions of variables $i$, $j$, and $k$; however, multiplication of $i, j$, and $k$ does not commute.

This definition does not meet contemporary standards of rigor, but is much more intuitive.
2.3. Basic Properties. Equation 2.3 immediately defines the multiplication of distinct pairs of $i$, $j$, and $k$:

\[
\begin{align*}
ij &= k & ji &= -k \\
jk &= i & kj &= -i \\
ki &= j & ik &= -j
\end{align*}
\]

These rules are consistent with the full definition of the product of quaternions given in equation 2.2.

If $q = a + bi + cj + dk$, define the real part of $q$ as $\text{Re}(q) = a$. Likewise, the imaginary part is $\text{Im}(q) = bi + cj + dk$.

The conjugate of $q = a + bi + cj + dk$ is $\bar{q} = a - bi - cj - dk$. For quaternions $q$ and $r$, the conjugate has the property:

\[
\bar{qr} = (\bar{r})(\bar{q})
\]

The norm of $q$ is $\|q\| = q\bar{q} = a^2 + b^2 + c^2 + d^2$ and the inverse $q^{-1}$, satisfying $qq^{-1} = q^{-1}q = 1$, is $\frac{q}{\|q\|}$.

Consider quaternions with no real part $v_1 = x_1i + y_1j + z_1k$ and $v_2 = x_2i + y_2j + z_2k$. Their product is

\[
v_1v_2 = (-x_1x_2 - y_1y_2 - z_1z_2) + (y_1z_2 - z_1y_2)i + (z_1x_2 - x_1z_2)j + (x_1y_2 - y_1x_2)k
\]

If $v_1$ and $v_2$ are interpreted as vectors $v_1$ and $v_2$ such that $i$, $j$, and $k$ are $x$, $y$, and $z$ unit vectors, respectively, then

\[
v_1v_2 = -v_1 \cdot v_2 + v_1 \times v_2
\]

This is the link between the quaternion product and vector products. It serves a crucial role in the derivation of quaternion rotation in Section 3.

2.4. Quaternions as Vector Spaces. It is easy to see that $\mathbb{H}$ is a four-dimensional vector space over $\mathbb{R}$: quaternions add and scale just as in $\mathbb{R}^4$.

Note that for quaternions $q_0$, $q_1$, and $q_2$ and real number $\alpha$ the following are true:

\[
q_0(\alpha q_1) = \alpha(q_0q_1)
\]

\[
q_0(q_1 + q_2) = q_0q_1 + q_0q_2
\]

This establishes that multiplication on the left by $q_0$ is a linear transformation (over $\mathbb{R}$). The matrix for left-multiplication by $a + bi + cj + dk$, with respect to the basis $(1, i, j, k)$, is

\[
\begin{pmatrix}
a & -b & -c & -d \\
b & a & -d & c \\
c & d & a & -b \\
d & -c & b & a
\end{pmatrix}
\]

This matrix is directly derived from the multiplication formula 2.2. If it is multiplied to a column vector representing another quaternion (also with respect to the basis $(1, i, j, k)$), the resulting column vector represents the product of the quaternions.

The transpose of the matrix simply changes the sign of $b$, $c$, and $d$, so it corresponds to the conjugate of the quaternion. Right multiplication has a similar matrix representation.
Quaternions also may also be represented as a two-dimensional vector space over \( \mathbb{C} \). With respect to the basis \((1, j)\), the coordinate vector of \( a + bi + cj + dk \) is

\[
\begin{pmatrix}
a + bi \\
c + di
\end{pmatrix}
\]

In this interpretation, multiplication is still a linear transformation, with matrix

\[
\begin{pmatrix}
a + bi & c + di \\
-c + di & a - bi
\end{pmatrix}
\]

2.5. **Algebraic Properties.** Matrix 2.12 is unitary — its conjugate transpose is also its inverse. Note that this is the form of an arbitrary unitary matrix with determinant 1; given the top row, the bottom row is forced. This means that the set of unit quaternions (quaternions with norm 1) is isomorphic to the special unitary group SU(2).

\( \mathbb{H} \) is classified as a **associative division algebra.** An associative algebra is a vector space with an associative multiplication that distributes over addition. A division algebra satisfies the property that for any \( x \) and nonzero \( y \), there are \( q_1 \) and \( q_2 \) such that \( x = q_1 y \) and \( x = yq_2 \). For quaternions, the quotients may simply be found using \( y^{-1} \): \( q_1 = xy^{-1} \) and \( q_2 = y^{-1}x \).

In 1878, 12 years after Hamilton’s death, Ferdinand Georg Frobenius proved that the only finite-dimensional division algebras over the real numbers are the \( \mathbb{R} \), \( \mathbb{C} \), and \( \mathbb{H} \). The crux of the argument deals with orthonormal generators for the subspace of pure imaginary elements of each set. Those are elements that satisfy the following properties:

1. Their squares are negative real numbers.
2. Their norms are 1.
3. Any pure imaginary can be produced in terms of them using scaling, addition, and multiplication.
4. Each one cannot be expressed in terms of the others using scaling, addition, and multiplication.
5. For any pair \( e_1 \) and \( e_2 \), \( \text{Re}(e_1 e_2) = 0 \) (they are normal with respect to that inner product).

For \( \mathbb{R} \), there are no such elements. For \( \mathbb{C} \), the only one is a choice of \( \pm i \). For \( \mathbb{H} \), there are two: a choice of two of \( i, j, \) and \( k \), negative or positive. The third may be generated as the product of the chosen two. Frobenius showed that there cannot be more than two elements that satisfy all 5 properties. [6]

3. **Quaternionic Rotation**

Three-dimensional rotation has applications in computer graphics and aeronautical navigation. The rotation of a single vector around the origin merely amounts to transforming it to a different vector of the same magnitude, so its usefulness may not be readily apparent. Rotations are most useful when applied to a set of vectors, such the vertices of an object model.

In his search for the algebra of triples, Hamilton wanted more than just a well-defined algebra. He sought an analogue for the isomorphism between complex multiplication and planar rotation. Such an analogue does exist. Here, we will derive the rotation formula in terms of vectors, and using the link between the
quaternion product of pure imaginaries and vector products, produce a quaternion formula.

A three-dimensional rotation of a vector $v$ is done by an angle $\theta$ about a normal axis $n$. In order to produce a formula for the rotated vector $u$, decompose $v$ into the sum of a part parallel to $n$, $v_{\parallel}$, a perpendicular part $v_{\perp}$. $v_{\parallel}$ is simply the projection of $v$ onto $n$, so we have

$$v_{\parallel} = (n \cdot v)n$$

(3.1)

$$v_{\perp} = v - (n \cdot v)n$$

Rotation of $v$ about $n$ does not alter $v_{\parallel}$ and simply rotates $v_{\perp}$ in the plane normal to $n$. This reduces the problem to finding $u_{\perp}$ by planar rotation.

If $w$ is normal to $v_{\perp}$ and lies on the plane normal to $n$, then

$$u_{\perp} = \cos \theta v_{\perp} + \sin \theta w$$

So what is $w$? It is normal to $n$ and $q_{\perp}$ and has the same magnitude as $v_{\perp}$. Given that $n$ is normal and forms a right angle with $v_{\perp}$, $w = n \times v_{\perp}$. This can be simplified even further:

$$w = n \times (v - v_{\parallel}) = n \times v - n \times v_{\parallel} = n \times v$$

(3.2)

We are now equipped to construct a formula for $u_{\perp}$, and, by adding $v_{\parallel}$, the fully rotated $u$. [7]

$$u = \cos \theta (v - (n \cdot v)n) + \sin \theta (n \times v) + (n \cdot v)n$$

(3.3)
Now our task is to somehow transform this into a formula of quaternions. The trick is in switching to half-angles:
\[
(3.4) \quad u = (\cos \frac{\theta}{2} - \sin \frac{\theta}{2})(v - (n \cdot v)n) + \sin \frac{\theta}{2}\cos \frac{\theta}{2}(n \times v) + (n \cdot v)n
\]

Regrouping the terms by trigonometric function:
\[
(3.5) \quad u = \cos \frac{\theta}{2}v + 2\sin \frac{\theta}{2}\cos \frac{\theta}{2}(n \times v) - \sin \frac{\theta}{2}(v - 2(n \cdot v)n)
\]

Now we change the vector products into the quaternion products: cross product minus dot product. Here, two adjacent vectors denote their quaternion product.
\[
(3.6) \quad u = \cos \frac{\theta}{2}v + 2\sin \frac{\theta}{2}\cos \frac{\theta}{2}(nv - vn) - \sin \frac{\theta}{2}nvn
\]

Finally, we separate out terms to arrive at the quaternion formula for rotation of \(v\) about normal axis \(n\) by angle \(\theta\). [8]
\[
(3.7) \quad u = (\cos \frac{\theta}{2} + \sin \frac{\theta}{2}n)v(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}n)
\]

Calling \(q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2}n\), we have \(u = qv\bar{q}\). Since \(n\) is normal, \(q\) is normal:
\[
\| \cos \frac{\theta}{2} + \sin \frac{\theta}{2}(b + cj + dk) \| = \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}(b^2 + c^2 + d^2)
\]
\[
= \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}(b^2 + c^2 + d^2)
\]
\[
= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} = 1
\]

Since \(q\bar{q} = \|q\| = 1, \bar{q} = q^{-1}\). Furthermore, any normal quaternion can be expressed in the form of \(q\). Therefore every normal quaternion represents a three-dimensional rotation.

The conjugate of a \(q\) represents the rotation by \(-\theta\) about \(n\). This is illustrated by the fact that the same rotation is produced by rotating by \(\theta\) about \(-n\). \(-q\) represents the same rotation as \(q\). These facts imply that every rotation determines two quaternions, which are opposite each other.

Multiplying normal quaternions represents the composition of the rotations they represent:
\[
(3.9) \quad (rq)v\bar{r} = rqv\bar{q}r = r(qv\bar{q})\bar{r}
\]

The non-commutativity of quaternions accurately reflects the non-commutativity of three-dimensional rotations.

Defining the exponential function to be its Taylor series
\[
(3.10) \quad \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\]
allows it to be well-defined on quaternions. Interestingly, Euler’s formula has a quaternion analogue,
\[
(3.11) \quad \exp(\theta n) = \cos(\theta) + \sin(\theta)n
\]
where \( n \) is any pure imaginary unit quaternion. This makes for a nice similarity between planar rotation with complex numbers and three-dimensional rotation with quaternions, with the peculiarity that quaternions use half-angles.

4. **Rotation Interpolation with Quaternions**

A quaternion may represent the static orientation of an object in 3-dimensional space. This is accomplished by interpreting the orientation as a rotation from a chosen canonical orientation. The quaternion for the orientation is simply the quaternion for that rotation. This representation is particularly useful for smooth interpolation between orientations. To understand why, we must first look at the most common representation of orientation, Euler angles.

4.1. **Euler Angles.** Rotation in of a vector in the plane by an angle \( \theta \) is given by the linear transformation with matrix

\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]

In three-dimensional space, rotation about the \( x \)-axis is simply given by applying the same transformation in the \( yz \)-plane:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix}
\]

Rotation about the \( y \)- and \( z \)-axes have matrices produced in the same manner.

Euler’s Rotation Theorem gives that any three-dimensional rotation may be described in terms of at most three rotations about the \( x \), \( y \), and \( z \) axes. Conceptualizing how this works is aided by two ideas. First, every rotation is about some axis, which may be expressed as a unit vector. Second, any point on the unit sphere may be expressed in terms of longitude and latitude. So one rotation is about the desired axis, the other two determine the latitude and longitude of the axis. This method lends itself to an easy method of computing the image of vectors under rotation: composing three matrices like matrix 4.2 yields a full three-dimensional rotation.

![Figure 3. Rotation with Euler Angles.](image)

The first rotation \( \phi \), about the vertical \( z \)-axis, determines the longitude of the desired axis of rotation. The second rotation \( \theta \) is applied about the rotated \( x \)-axis and determines the latitude. This rotation, applied to the \( z \)-axis, produces the desired axis of rotation. The final rotation \( \psi \) is about this axis. [9]
4.2. **Gimbal Lock.** A gimbal is a set of concentric rings with an object in the center. The rings and object are connected by alternating axes. When there are three or more rings, the object may be placed in any orientation, while keeping the outer ring still. When there are three rings, a gimbal represents rotation by Euler angles: the angle of the middle ring with respect to outer ring is $\phi$, the angle of the inner ring with the middle ring is $\theta$, and the angle of the object with the inner ring is $\psi$.

![Gimbal](https://via.placeholder.com/150)

**Figure 4.** Gimbal, by Svjo (Own work), distributed under a CC BY-SA 4.0 license, via Wikimedia Commons.

Gimbal lock is a phenomenon that occurs when two rings line up. Rotating about two different axes will rotate the object about the same axis. Since there is only one remaining independent way to rotate the object, a degree of freedom has been lost. The gimbal is not actually locked — any possible orientation of the object is still achievable. What has been lost is the ability to continuously turn the object about the axis normal to the two available axes. [10]

4.3. **Application Case Study: Camera.** In 3D graphics, the perspective from which the scene is viewed is represented as a camera object in the world model. The camera naturally has an orientation that is tracked in the model. It is also natural to expect that we should want to move the camera continuously between two orientations — that is, continuously interpolate its orientation. Furthermore, we should want the front of the camera object, which represents the perspective, to follow a geodesic path (the shortest possible path between two points on the sphere).

Consider the case where the orientation of the camera is represented by Euler angles $\theta$, $\phi$, and $\psi$. If $\text{Rot}(\alpha, \mathbf{n})$ denotes a rotation by $\alpha$ about $\mathbf{n}$, the orientation is $\text{Rot}(\theta, -\mathbf{e}_z) \circ \text{Rot}(\phi, -\mathbf{e}_y) \circ \text{Rot}(\psi, \mathbf{e}_z)$ from the canonical orientation. This is different from the earlier example of Euler angles; it is one of many equally valid choices of axes. With this choice of axes, and choosing the direction of the perspective to be $\mathbf{p} = (1, 0, 0)$ in the canonical orientation, $\theta$, $\phi$, and $\psi$ represent the yaw, pitch, and roll of the front of the camera, respectively. Increasing $\theta$ turns the perspective right, increasing $\phi$ turns the perspective up, and increasing $\psi$ rotates the perspective clockwise.
Continuously changing the angles will continuously change the orientation. Linearly changing the angles will not necessarily make \( p \) follow a geodesic path, however. This may be fixed by simply changing different angles at changing rates. However, there is a problem if the camera is at an orientation at which gimbal lock occurs. Say \( \theta = 0, \phi = \frac{\pi}{2}, \) and \( \psi = 0; \) at this orientation \( p = (0, 0, 1). \) Changing \( \theta \) and \( \psi \) at this orientation will not change the direction of \( p; \) they both rotate the perspective. That means that it is impossible to tilt the perspective to the side without straying from the geodesic path or discontinuously changing the angles. In some applications, this is limitation is disregarded. In most first-person video games there is rarely a reason the player would look straight up or down. However, if total freedom of movement is desired, Euler angles are not a good way to represent orientation.

4.4. The Quaternion Solution. The quaternion representation of the same orientation of looking straight up is \( q_0 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} j; \) the perspective has been rotated by \( \frac{\pi}{2} \) about \((0, -1, 0).\) Say that we want to tilt the perspective to the left by angle \( \alpha \) over time interval \( t: [0, 1]. \) The orientation at time \( t \) is simply

\[
r_t q_0 = (\cos(\frac{\alpha}{2} t) + \sin(\frac{\alpha}{2} t)i)q_0
\]

The axis, \((1, 0, 0),\) was defined to be normal to the perspective \( q_0 p \bar{p}_0, \) so the perspective will naturally follow a geodesic path. Another nice property of this parametrization is that the angular velocity of any vector under the changing orientation is constant. \([10]\)

We can’t take for granted that we can produce an axis that is normal to \( q_0 p \bar{p}_0. \) What if we want to interpolate the orientation between two specific orientations \( q_0 \) and \( q_1? \) Producing an axis \( n \) that is normal to both \( q_0 p \bar{p}_0 \) and \( q_1 p \bar{q}_1, \) as well as the angle \( \alpha \) between them, may be done by taking their product, which yields

\[
\text{cos}(\alpha) + \text{sin}(\alpha)n
\]

If the interpolation will be done in \( N \) discrete steps, we may initially compute

\[
r_{1/N} = \cos(\frac{\alpha}{2N}) + \text{sin}(\frac{\alpha}{2N})n
\]

and for each step, multiply the orientation by \( r_{1/N}. \) This is equivalent to rotating the orientation by \( \frac{\alpha}{N} \) each step.

A fewer-dimensional analogy for the difference between using Euler angles and quaternions for representing orientation is the representation of points on the unit sphere. Euler angles correspond to the angles of longitude and latitude, and quaternions correspond to the explicit coordinates of the point. The analogue to orientations at which gimbal lock occur is the poles of the sphere. Just as changing the longitude does not affect a point at a pole, changing the three Euler angles will only change the orientation in two ways. Unfortunately, there is no analogue for the problems caused by gimbal lock. They are unique to higher dimensions.

Conclusion

Quaternions, at first glance, seem to be a peculiar system; more of a novelty than a practical concept. However, their historical importance is undeniable. Quaternions are the predecessor to vectors, an indispensable concept in modern mathematics and physics. Their importance is more than historical, though. Frobenius’s
Theorem asserts a sort of justification of their algebraic rules, and they have a concrete advantage over Euler angles for describing rotations. These facts are evidence that quaternions are more than just relics.

References