Chaos and Computation of Lyapunov Exponents

Bryan Garcia

AMS 114, Fall 2018

December 11, 2018
Outline

1. Introduction
   - Dynamics and Chaos in One-Dimensional Systems
   - Lyapunov Exponents

2. Computing Lyapunov Exponents for N-Dimensional Dynamical Systems
   - An Overview of the Method
   - Implementing the Algorithm
   - Lyapunov Spectra Results for Hénon, Lorenz, and Rössler Systems

3. Computation of Maximal Lyapunov Exponent from Time-Series Data
   - Phase-Space Reconstruction
   - Implementing the Algorithm
Outline

1 Introduction
   - Dynamics and Chaos in One-Dimensional Systems
   - Lyapunov Exponents

2 Computing Lyapunov Exponents for N-Dimensional Dynamical Systems
   - An Overview of the Method
   - Implementing the Algorithm
   - Lyapunov Spectra Results for Hénon, Lorenz, and Rössler Systems

3 Computation of Maximal Lyapunov Exponent from Time-Series Data
   - Phase-Space Reconstruction
   - Implementing the Algorithm
Before we talk about Lyapunov Exponents...

Some things to discuss:
Before we talk about Lyapunov Exponents...

Some things to discuss:

- What are Dynamical Systems?
Before we talk about Lyapunov Exponents...

Some things to discuss:

- What are Dynamical Systems?
- What is Chaos?
Before we talk about Lyapunov Exponents...

Some things to discuss:

- What are Dynamical Systems?
- What is Chaos?
- What are some examples of both?
Before we talk about Lyapunov Exponents...

Some things to discuss:

- What are Dynamical Systems?
- What is Chaos?
- What are some examples of both?
- How can we quantify this sensitivity?
The Logistic Map

Our equation:

\[ x_{n+1} = rx_n(1 - x_n) \]
The Logistic Map

Our equation:

\[ x_{n+1} = r x_n (1 - x_n) \]

- Why bring it up?
Visualization: Sensitivity to Change

Figure: $x_0 = 0.9, r = 3.6$.

Figure: $x_0 = 0.9, r = 3.9$. 
The Lyapunov Exponent

Some things to discuss:
The Lyapunov Exponent

Some things to discuss:

- What is a Lyapunov Exponent?
The Lyapunov Exponent

Some things to discuss:

- What is a Lyapunov Exponent?
- Where does it come from?
The Lyapunov Exponent

Some things to discuss:

- What is a Lyapunov Exponent?
- Where does it come from?

Equation for Lyapunov Exponent
The Lyapunov Exponent

Some things to discuss:

- What is a Lyapunov Exponent?
- Where does it come from?

Equation for Lyapunov Exponent

\[
\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \log_2 \left( \frac{E(i)}{E(i-1)} \right)
\]
The Lyapunov Exponent

Some things to discuss:
- What is a Lyapunov Exponent?
- Where does it come from?

Equation for Lyapunov Exponent

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \log_2 \left( \frac{E(i)}{E(i-1)} \right)$$

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log_2 (||f'(x(i))||)$$
Visualization: The Lyapunov Exponent

Lyapunov Exponent Plot for Logistic Map

$r$

$\lambda_1$
Visualization: The Lyapunov Exponent

**Figure:** Blue: $x_0 = 0.9$; Red: $x_0' = 0.9001$; $r = 3.2359$
Outline

1. Introduction
   - Dynamics and Chaos in One-Dimensional Systems
   - Lyapunov Exponents

2. Computing Lyapunov Exponents for N-Dimensional Dynamical Systems
   - An Overview of the Method
   - Implementing the Algorithm
   - Lyapunov Spectra Results for Hénon, Lorenz, and Rössler Systems

3. Computation of Maximal Lyapunov Exponent from Time-Series Data
   - Phase-Space Reconstruction
   - Implementing the Algorithm
Our New Setting
Our New Setting

Lyapunov Spectrum (re)Defined

\[ \lambda_i = \lim_{t \to \infty} \log_2 \left( \frac{p_i(t)}{p_i(0)} \right) \]
Our New Setting

Linearized Equations and Vector Frame Propogation

\[
\begin{pmatrix}
\delta X_{n+1} \\
\delta Y_{n+1}
\end{pmatrix}
= J(x_n, y_n)
\begin{pmatrix}
\delta X_n \\
\delta Y_n
\end{pmatrix}
\]
Our New Setting

Linearized Equations and Vector Frame Propogation

\[
\begin{pmatrix}
\delta X_{n+1} \\
\delta Y_{n+1}
\end{pmatrix} = J(x_n, y_n) \begin{pmatrix}
\delta X_n \\
\delta Y_n
\end{pmatrix}
\]
Algorithm Outline

- Define initial conditions.
- Integrate non-linear equations to obtain a portion of our fiducial trajectory.
Algorithm Outline

- Obtain the Jacobian matrix of our linearized equations and apply the matrix to our vector frame.
- Update orbital divergence values from our new set of vectors.
Some Issues!

- Divergence of Vectors.
- Loss of Orientation.
- Computer Limitations.
Visualization of said issues

**Figure:** Hénon map illustrating propagation of vectors.
Algorithm Outline

- Apply Gram-Schmidt to obtain a new orthonormal vector frame.
- Repeat our procedure along entirety of fiducial trajectory.
- Compute $\lambda_i$ by dividing orbital divergence by number of iterations.
Computing Spectra

Hénon Map

\[ X_{n+1} = 1 - aX_n^2 + Y_n \]
\[ Y_{n+1} = bX_n \]

Linearized Hénon Map

\[ J(x_n, y_n) = \begin{pmatrix} -2ax_n & 1 \\ b & 0 \end{pmatrix} \]

Published Results:

- Parameter Values: \( a = 1.4, b = 0.3 \)
- Lyapunov Spectrum:
  \( \lambda_1 = 0.603, \lambda_2 = -2.34 \)

My Results:

- Lyapunov Spectrum:
  \( \lambda_1 = 0.6015, \lambda_2 = -2.3385 \)

**Figure:** Number of Iterations: \( n = 1000 \)
Computing Spectra

Lorenz Equations
\[
\begin{align*}
\dot{X} &= \sigma (Y - X) \\
\dot{Y} &= X (\rho - Z) - Y \\
\dot{Z} &= XY - \beta Z
\end{align*}
\]

Linearized Lorenz Equations
\[
J(\chi, \gamma, \zeta) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & -X \\ Y & X & -\beta \end{pmatrix}
\]

Published Results:
- Parameter Values: \( \sigma = 16, \rho = 45.92, \beta = 0.4 \)
- Lyapunov Spectrum: \( \lambda_1 = 2.16, \lambda_2 = 0, \lambda_3 = -32.4 \)

My Results:
- Lyapunov Spectrum: \( \lambda_1 = 2.0686, \lambda_2 = -0.0115, \lambda_3 = -32.3537 \)

Figure: \( h = 0.001, t = [0,100] \)
Computing Spectra

**Rössler Equations**

\[ \dot{X} = -(Y + Z) \]
\[ \dot{Y} = X + aY \]
\[ \dot{Z} = b + Z(X - c) \]

**Linearized Rössler Equations**

\[ J(x, y, z) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ Z & 0 & X - c \end{pmatrix} \]

**Published Results:**

- **Parameter Values:**
  \[ a = 0.15, \quad b = 0.20, \quad c = 10 \]
- **Lyapunov Spectrum:**
  \[ \lambda_1 = 0.13, \quad \lambda_2 = 0, \quad \lambda_3 = -14.1 \]

**My Results:**

- **Lyapunov Spectrum:**
  \[ \lambda_1 = 0.1258, \quad \lambda_2 = 0.0047, \quad \lambda_3 = -14.1455 \]

**Figure:** \( h = 0.01, \quad t = [0,1000] \)
Outline

1 Introduction
   - Dynamics and Chaos in One-Dimensional Systems
   - Lyapunov Exponents

2 Computing Lyapunov Exponents for N-Dimensional Dynamical Systems
   - An Overview of the Method
   - Implementing the Algorithm
   - Lyapunov Spectra Results for Hénon, Lorenz, and Rössler Systems

3 Computation of Maximal Lyapunov Exponent from Time-Series Data
   - Phase-Space Reconstruction
   - Implementing the Algorithm
Our New Setting

With only a time-series...

- Phase-space plus tangent-space approach fails.
- No equations to work with.
Reconstruction with Time-Delay Embedding

A Result of Takens’ Theorem
Reconstruction with Time-Delay Embedding

A Result of Takens’ Theorem
Given a time series \( x(t) \), we can define an \((n+1)\) dimensional embedding as:

\[
[x(t), x(t + \tau), \ldots, x(t + n\tau)]
\]
A Quick Visualization of Reconstruction
Summary of MLE Algorithm with Illustration

Figure: Sketch of MLE Algorithm
MLE Algorithm Results

Parameters of System

- $\sigma = 16$, $\rho = 45.92$, $\beta = 0.4$
- Initial Lorenz Conditions: $\bar{x} = (0.2, 0.3, 0.5)$
- RK4 Time-Step: $h = 0.01$
- RK4 Integration Time: $[0 : h : 100]$
- Time-Delay Parameter: $\tau = 30$
- Fixed-Time-Step: $N = 100$ (1 second).
- $\text{distMin} = 0.001$
- $\text{distMax} = 10$
- $\text{angleMax} = \frac{\pi}{6}$

Published Paper Results for MLE:
- $\lambda_{max} = 2.16$

Implementation Results for MLE
- $\lambda_{max} = 2.305$
Questions