Stratification in tensor triangular geometry with applications to spectral Mackey functors

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based on joint work with

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- Technically, this amounts to a classification of the thick triangulated tensor-ideal subcategories ("thick $\otimes$-ideals") of $\mathcal{K}$
- $X, Y \in \mathcal{K}$ are equivalent iff $\langle X \rangle = \langle Y \rangle$

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- Example: The classification theorem for $\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}$ is in terms of a notion of support for finite spectra defined using Morava K-theories.
- Greatly clarified by Balmer who introduced the *universal* notion of support for essential small tensor-triangulated categories.

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- This universal notion of support classifies the thick $\otimes$-ideals of $\mathcal{K}$:

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  e.g. $\langle X\rangle = \langle Y\rangle$ iff $\mathrm{supp}(X) = \mathrm{supp}(Y)$.

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$$\{\text{localizing } \otimes\text{-ideals of } \mathrm{D}(R)\} \xrightarrow{\cong} \{\text{subsets of } \mathrm{Spec}(R)\}$$

- In general, the statement fails strongly for R non-noetherian.

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- Developed further and applied to the problem of classifying localizing tensor-ideals by [Stevenson '13].

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Assume that $\mathrm{Spec}(\mathcal{T}^c)$ is a weakly noetherian space.

We have a notion of support

$$\mathrm{Supp}(t) \subset \mathrm{Spec}(\mathcal{T}^c)$$

defined for each object $t \in \mathcal{T}$.

For compact $x \in \mathcal{T}^c$, $\mathrm{Supp}(x) = \mathrm{supp}(x)$.

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3. Apply the theory to new examples, notably in homotopy theory.

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2. (Minimality at all points) For each $\mathcal{P} \in \mathrm{Spec}(\mathcal{T}^c)$, the localizing $\otimes$-ideal $g(\mathcal{P}) \otimes \mathcal{T} = \mathrm{Loc}_{\otimes}\langle g(\mathcal{P})\rangle$ is a minimal localizing $\otimes$-ideal.

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That is, stratification is *equivalent* to the classification of localizing $\otimes$-ideals.

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Theorem. If $\mathrm{Spec}(\mathcal{T}^c)$ is *noetherian* then the local-to-global principle holds for $\mathcal{T}$.

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2. We obtain a complete description of the Bousfield lattice of $\mathcal{T}$. In particular, there is no difference between localizing tensor-ideals and Bousfield classes.
3. The telescope conjecture holds for $\mathcal{T}$ provided $\mathrm{Spec}(\mathcal{T}^c)$ is "generically noetherian".

Features of our theory of stratification

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Example: Reduces the problem of stratification to the *local categories* $\mathcal{T}_{\mathcal{P}} := \mathcal{T} / \operatorname{Loc}_{\otimes}\langle \mathcal{P} \rangle$.

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  $$\varphi : \mathrm{Spec}(\mathcal{D}^c) \to \mathrm{Spec}(\mathcal{C}^c)$$

  denote the induced map. If $\mathcal{P} \in \mathrm{Spec}(\mathcal{D}^c)$ is a point such that $\varphi^{-1}(\{\varphi(\mathcal{P})\}) = \{\mathcal{P}\}$ then minimality at $\mathcal{P}$ in $\mathcal{D}$ implies minimality at $\varphi(\mathcal{P})$ in $\mathcal{C}$.

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- It allows you to check stratification, in some circumstances, after a finite étale extension.

Permanence

- There is also a weak form of finite étale descent, which in the interests of time I will gloss over:

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- An important ingredient for our equivariant applications.

Generality

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This theory provides a uniform perspective on old and new classification theorems.

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- None of the above categories are canonically stratified in the sense of BIK.

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Theorem. Let G be a finite group and let $\mathbb{E}$ be a commutative ring spectrum such that $\mathrm{Spec}(\mathrm{D}(\mathbb{E})^c)$ is noetherian. If $\mathrm{D}(\mathbb{E})$ is stratified, then so is the category $\mathrm{Mack}_G(\mathbb{E})$ of spectral G-Mackey functors with coefficients in $\mathbb{E}$.

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1. It is a property of the category $\mathcal{T}$ whether it is stratified. In this case, the classification of localizing $\otimes$-ideals is given in terms of the *set* underlying the Balmer spectrum $\mathrm{Spec}(\mathcal{T}^c)$.
2. We can then try to determine the *topology* of $\mathrm{Spec}(\mathcal{T}^c)$, which would result in the classification of thick $\otimes$-ideals of $\mathcal{T}^c$.

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In this sense, BIK-stratification is both stronger and weaker than our notion of stratification.

Thanks for listening!

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