Stratification in tensor triangular geometry with applications to spectral Mackey functors

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based on joint work with

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Wild classification problems

• Classify all finite-dimensional representations of most finite groups in positive characteristic
• Classify all finite CW-complexes up to homotopy equivalence
• ···
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- We work *stably*, that is, in a suitable stable homotopy category $\mathcal{K}$ of such objects, and
- We regard two objects as equivalent if they can be built from each other using the *tensor-triangulated structure* of the stable category.
- Technically, this amounts to a classification of the thick triangulated tensor-ideal subcategories ("thick $\otimes$-ideals") of $\mathcal{K}$
- $X, Y \in \mathcal{K}$ are equivalent iff $\langle X \rangle = \langle Y \rangle$
Historical examples

• $K = \text{stmod}(kG)$ the stable module category of finite-dimensional $k$-linear representations of a finite group $G$ (Benson–Carlson–Rickard '97)

• $K = \text{SH}_{\text{fin}}$ the category of finite spectra (Hopkins–Smith '98)

• $K = \text{D}_{\text{perf}}(X)$ the derived category of perfect complexes of a scheme $X$ (Thomason '97)
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- Example: The classification theorem for $\mathcal{K} = \text{stmod}(kG)$ uses the theory of *support varieties* which assigns to each finite-dimensional representation $X$ a certain closed subscheme $V_G(X) \subseteq \text{Proj}(H^*(G, k))$.

- Example: The classification theorem for $\mathcal{K} = \text{SH}_{\text{fin}}$ is in terms of a notion of support for finite spectra defined using Morava $K$-theories.

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The spectrum of a tt-category (Balmer ’05)

- **Universal property:** $(\text{Spec}(K), \text{supp})$ is the universal space with a well-behaved notion of support for objects of $K$.
- This universal notion of support classifies the thick $\otimes$-ideals of $K$: 
  
  \[
  \text{Spec}(K) := \{ P \subseteq K \mid P \text{ is a prime tt-ideal} \}
  \]
  
  \[
  \text{supp}(X) := \{ P \in \text{Spec}(K) \mid X \not\in P \}
  \]
  
  - $X \in K$: object $\Rightarrow \text{supp}(X) \subseteq \text{Spec}(K)$ closed subset.
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\( \mathcal{K} \) essentially small tt-category \( \rightsquigarrow \text{Spec}(\mathcal{K}) \) topological space

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$X \in \mathcal{K}$ object $\rightsquigarrow$ supp($X$) $\subseteq$ Spec($\mathcal{K}$) closed subset
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The spectrum of a tt-category (Balmer ’05)

Theorem (Abstract Classification Theorem).

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\text{under mild hypotheses on } K, \text{ there is a bijection } \\
\{\text{thick } \otimes \text{-ideals of } K\} \rightarrow \{\text{Thomason subsets of Spec}(K)\} \\
\mapsto \bigcup \text{support}(X) \\
\text{a Thomason subset := a union of closed subsets each of which has quasi-compact complement} \\
e.g. \text{if Spec}(K) \text{ is noetherian then "Thomason" = "union of closed sets" = "specialization-closed"}
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\[\langle X \rangle = \langle Y \rangle \iff \text{support}(X) = \text{support}(Y)\]
The spectrum of a tt-category (Balmer ’05)

Theorem (Abstract Classification Theorem). Under mild hypotheses on $\mathcal{K}$, there is a bijection

$$\left\{\text{thick } \otimes\text{-ideals of } \mathcal{K}\right\} \sim \left\{\text{Thomason subsets of } \text{Spec}(\mathcal{K})\right\}$$

$$c \mapsto \bigcup_{X \in c} \text{supp}(X)$$

• a Thomason subset := a union of closed subsets each of which has quasi-compact complement
• e.g. If $\text{Spec}(\mathcal{K})$ is noetherian then "Thomason" = "union of closed sets" = "specialization-closed"
• Classifying the tt-ideals of $\mathcal{K}$ ↔ Describing the space $\text{Spec}(\mathcal{K})$
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The spectrum of a tt-category

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e.g. $\langle X \rangle = \langle Y \rangle$ iff supp($X$) = supp($Y$).
The spectrum of a tt-category

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Examples:

• Spec(D_{perf}(R)) = Spec(R) for any commutative ring R

• Spec(D_{perf}(X)) = X for any non-pathological scheme X

• Spec(stmod(kG)) = Proj(H^{\ast}(G, k)) for any finite group G

• ···
The spectrum of a tt-category (Balmer ’05)

Examples:

- $\text{Spec}(\text{D}^{\text{perf}}(R))$ for any commutative ring $R$
- $\text{Spec}(\text{D}^{\text{perf}}(\mathcal{X})) \cong \mathcal{X}$ for any non-pathological scheme $\mathcal{X}$
- $\text{Spec}(\text{stmod}(kG)) \cong \text{Proj}(H^\ast(G, k))$ for any finite group $G$
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• \( \ldots \)
The spectrum of a tt-category (Balmer ’05)

- Nice, but only classifies “compact” objects
  - e.g. finite-dimensional representations
  - finite spectra
  - perfect complexes
  - Often $K$ arises as the subcategory of compact objects $T_c \subset T$
    in a larger “rigidly-compactly generated tt-category” $T$.
  - e.g. $\text{stmod}(kG) \subset \text{StMod}(kG)$
  - $\text{SH}_{\text{fin}} \subset \text{SH}$
  - $\text{D}_{\text{perf}}(R) \subset \text{D}(R)$
  - Many of the most interesting objects are non-compact!
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  (e.g., the objects that represent cohomology theories)
What about the “big” objects?

• Want a similar understanding of the “big” objects of $T$.
• Technically, we would like to classify the localizing tensor-ideals of $T$.
• Example: Understanding the localizing tensor-ideals of $SH$ would provide a coarse classification of cohomology theories in algebraic topology.
• We can’t apply Balmer’s construction to $T$:
  1. $T$ is not essentially small
  2. The axioms for Balmer’s universal notion of support are not appropriate for big objects.
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1. Support for big objects is less well-behaved
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Let $R$ be a commutative noetherian ring. The usual cohomological support provides a bijection:

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- Develop a support theory for any compactly generated triangulated category $T$ equipped with an action by a noetherian graded commutative ring $R$.
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Think of \( \mathcal{T} \) as a "bundle" or "sheaf" of tt-categories over the space \( \text{Spec}( \mathcal{T}^c) \).

Assume \( \text{Spec}( \mathcal{T}^c) \) is weakly noetherian.

This means that each \( P \in \text{Spec}( \mathcal{T}^c) \) is "weakly visible" in the sense that \( \{P\} \) is the intersection of a Thomason subset and the complement of a Thomason subset:

\[ \{P\} = Y_1 \cap Y_2^c. \]

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Define \( g(P) := e_{Y_1} \otimes f_{Y_2} \in \mathcal{T} \).

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Intuitively, we can isolate $\mathcal{P}$ by a combination of a finite colocalization $e_{Y_1} \otimes -$ (which restricts to $Y_1$) and a finite localization $f_{Y_2} \otimes -$ (which restricts to $Y_2^c$).

Define $g(\mathcal{P}) := e_{Y_1} \otimes f_{Y_2} \in \mathcal{T}$. Think of

$$g(\mathcal{P}) \otimes \mathcal{T} = \text{Loc}_\otimes \langle g(\mathcal{P}) \rangle$$

as the “stalk” at $\mathcal{P}$.

For each object $t \in \mathcal{T}$, define

$$\text{Supp}(t) := \{ \mathcal{P} \in \text{Spec}(\mathcal{T}^c) \mid g(\mathcal{P}) \otimes t \neq 0 \} \subset \text{Spec}(\mathcal{T}^c)$$
Balmer–Favi big support (Balmer–Favi ’11, Stevenson ’13)

Assume that $\text{Spec}(\mathcal{T}^c)$ is a weakly noetherian space.

We have a notion of support

$$\text{Supp}(t) \subset \text{Spec}(\mathcal{T}^c)$$

defined for each object $t \in \mathcal{T}$.

For compact $x \in \mathcal{T}^c$, $\text{Supp}(x) = \text{supp}(x)$. 
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3. Apply the theory to new examples, notably in homotopy theory.
Stratification

Following BIK, we say $\mathcal{T}$ is *stratified* if the following two conditions hold:

1. (The local-to-global principle) Any object $t \in \mathcal{T}$ can be reconstructed from its germs $g(P) \otimes t$. In other words, $t \in \text{Loc} \otimes \langle g(P) \otimes t \mid P \in \text{Spec}(\mathcal{T}_c) \rangle$.

2. (Minimality at all points) For each $P \in \text{Spec}(\mathcal{T}_c)$, the localizing $\otimes$-ideal $g(P) \otimes \mathcal{T} = \text{Loc} \otimes \langle g(P) \rangle$ is a minimal localizing $\otimes$-ideal.
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Following BIK, we say $\mathcal{I}$ is *stratified* if the following two conditions hold:

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Following BIK, we say $\mathcal{T}$ is \textit{stratified} if the following two conditions hold:

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**Theorem.** Let $\mathcal{T}$ be a r.c.g. tt-category with $\text{Spec}(\mathcal{T}^c)$ weakly noetherian.
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**Theorem.** Let $\mathcal{T}$ be a r.c.g. tt-category with $\text{Spec}(\mathcal{T}^c)$ weakly noetherian. TFAE:

1. The local-to-global principle holds for $\mathcal{T}$ and we have minimality at each point $P \in \text{Spec}(\mathcal{T}^c)$.
2. The map $\{\text{localizing } \otimes \text{-ideals of } \mathcal{T}\} \xrightarrow{\text{Supp}} \{\text{subsets of } \text{Spec}(\mathcal{T}^c)\}$ defined by $L \mapsto \bigcup_{t \in L} \text{Supp}(t)$ is a bijection. That is, stratification is equivalent to the classification of localizing $\otimes$-ideals.
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The local-to-global principle does not always hold (Stevenson '14). However, we improve results of Stevenson to:

Theorem. If \( \text{Spec}(T_c) \) is noetherian then the local-to-global principle holds for \( T \).
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**Theorem.** If $\text{Spec}(\mathcal{T}^c)$ is *noetherian* then the local-to-global principle holds for $\mathcal{T}$. 
Consequences of stratification

Theorem. If $T$ is stratified then:

1. The Balmer–Favi notion of support satisfies the full tensor-product formula: $\text{Supp}(s \otimes t) = \text{Supp}(s) \cap \text{Supp}(t)$ for any $s, t \in T$.

2. We obtain a complete description of the Bousfield lattice of $T$. In particular, there is no difference between localizing tensor-ideals and Bousfield classes.

3. The telescope conjecture holds for $T$ provided Spec($T_c$) is "generically noetherian."
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Features of our theory of stratification

1. Universality
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Universality

In a slogan: "The Balmer–Favi notion of support provides the universal approach to stratification in weakly noetherian contexts."

Theorem. Let $\sigma : T \rightarrow P(X)$ be a support function for $T$ lying in a weakly noetherian space $X$. If this notion of support stratifies $T$ in a way compatible with the usual classification of thick $\otimes$-ideals of $T_c$, then there is a unique identification $(X, \sigma) \sim = (\text{Spec}(T_c), \text{Supp})$ with the Balmer–Favi notion of support.
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Corollary. If $\mathcal{T}$ is a rigidly-compactely generated tt-category which is stratified in the sense of BIK by the action of a graded-noetherian ring $R$, then the BIK space of supports $\text{supp}_R(\mathcal{T})$ is canonically homeomorphic to $\text{Spec}(\mathcal{T}^c)$.
Corollary. If \( \mathcal{T} \) is a rigidly-compactly generated tt-category which is stratified in the sense of BIK by the action of a graded-noetherian ring \( R \), then the BIK space of supports \( \text{supp}_R(\mathcal{T}) \) is canonically homeomorphic to \( \text{Spec}(\mathcal{T}^c) \) and the BIK notion of support coincides with the Balmer–Favi notion of support.
Permanence

This geometric theory of stratification exhibits good permanence properties under base-change functors. E.g., it satisfies versions of Zariski and étale descent.

Theorem. Suppose $T$ satisfies the local-to-global principle (e.g., suppose $\text{Spec}(T)$ noetherian).

If $\text{Spec}(T) = \bigcup_{i \in I} V_i$ is a cover by complements of Thomason subsets $V_i$.

Then $T$ is stratified if and only if each of the $t$-categories $T(V_i)$ is stratified.

Example: We can apply the theorem to a cover by quasi-compact open subsets (Stevenson).

Example: Reduces the problem of stratification to the local categories $T_P := T / \text{Loc} \otimes \langle P \rangle$. 
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Example: Reduces the problem of stratification to the local categories $\mathcal{T}_P := \mathcal{T}/\text{Loc}_{\otimes}\langle P \rangle$. 
There is also a weak form of finite étale descent, which in the interests of time I will gloss over:

Theorem. Let $F : C \rightarrow D$ be a finite étale morphism of rigidly-compactly generated tt-categories. Assume that both categories have noetherian spectrum, and let $\phi : \text{Spec}(D^c) \rightarrow \text{Spec}(C^c)$ denote the induced map. If $P \in \text{Spec}(D^c)$ is a point such that $\phi^{-1}(\{\phi(P)\}) = \{P\}$ then minimality at $P$ in $D$ implies minimality at $\phi(P)$ in $C$.

It allows you to check stratification, in some circumstances, after a finite étale extension.

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This theory provides a uniform perspective on old and new classification theorems.
Generality

**Theorem.** The following categories are stratified:

1. $\text{StMod}(kG)$ for a finite group $G$ and field $k$.
   - Due to Benson–Iyengar–Krause '11.
2. The derived category $\mathcal{D}_{\text{Qcoh}}(X)$ of a noetherian scheme $X$.
   - Due to Stevenson '13.
   - The resulting classification of localizing $\otimes$-ideals is originally due to Alonso Tarrío–Jeremías López–Souto Salorio '04.
3. The category of $E(n)$-local spectra for any $n \geq 0$.
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Equivariant stable homotopy theory

- We understand \( \text{Spec}(\text{SH}_G) \) as a set; interesting questions remain about its topology.

- \( \text{SH}_G \) is highly non-Noetherian (even for \( G = 1 \)) and the classification of localizing tensor-ideals is wide open.

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The Balmer spectrum of spectral Mackey functors

• We understand $\text{Spec}(\text{Mack}_G(E)_c)$ as a set in terms of $\text{Spec}(\text{D}(E)_c)$ for any commutative ring spectrum $E$ and finite group $G$.

• Interesting questions remain about the topology.

• For the category of derived $G$-Mackey functors ($E = \text{H}_\mathbb{Z}$), the spectrum has been completely computed by [Patchkoria–S.–Wimmer]. It captures precisely the height 0 and height $\infty$ chromatic layers of $\text{Spec}(\text{SH}_c G)$.

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Stratification for spectral Mackey functors

(Barthel–Heard–S. '21)

Theorem. Let $G$ be a finite group and let $E$ be a commutative ring spectrum such that Spec(D($E$) is noetherian. If $D(E)$ is stratified, then so is the category $\text{Mack}^G(E)$ of spectral $G$-Mackey functors with coefficients in $E$.

Corollary ($E = \text{H}_Z$): We obtain a classification of the localizing tensor-ideals of the category of derived $G$-Mackey functors.

Corollary ($E = L^nS$): We obtain a classification of the localizing tensor-ideals of the category of $E$($n$)-local spectral $G$-Mackey functors.
Theorem. Let $G$ be a finite group and let $E$ be a commutative ring spectrum such that $\text{Spec}(\text{D}(E)^c)$ is noetherian. If $\text{D}(E)$ is stratified, then so is the category $\text{Mack}_G(E)$ of spectral $G$-Mackey functors with coefficients in $E$. 


• Corollary ($E = H\mathbb{Z}$): We obtain a classification of the localizing tensor-ideals of the category of derived $G$-Mackey functors.

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**Theorem.** Let $G$ be a finite group and let $E$ be a commutative ring spectrum such that $\text{Spec}(D(E)^c)$ is noetherian. If $D(E)$ is stratified, then so is the category $\text{Mack}_G(E)$ of spectral $G$-Mackey functors with coefficients in $E$.

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**Theorem.** Let $G$ be a finite group and let $\mathbb{E}$ be a commutative ring spectrum such that $\text{Spec}(\mathcal{D}(\mathbb{E})^c)$ is noetherian. If $\mathcal{D}(\mathbb{E})$ is stratified, then so is the category $\text{Mack}_G(\mathbb{E})$ of spectral $G$-Mackey functors with coefficients in $\mathbb{E}$.

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- **Corollary ($E = L_n S$):** We obtain a classification of the localizing tensor-ideals of the category of $E(n)$-local spectral $G$-Mackey functors.
Our approach to understanding big tt-categories involves two steps:

1. It is a property of the category $\mathcal{T}$ whether it is stratified. In this case, the classification of localizing $\otimes$-ideals is given in terms of the set underlying the Balmer spectrum $\text{Spec}(\mathcal{T})$.

2. We can then try to determine the topology of $\text{Spec}(\mathcal{T})$, which would result in the classification of thick $\otimes$-ideals of $\mathcal{T}$. 
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In summary: Our notion of stratification based on the Balmer spectrum is rather flexible. It can be quite effective even when we only have partial knowledge of $\text{Spec}(T)$. This also distinguishes our notion of stratification from that of Benson–Iyengar–Krause: the latter simultaneously provides the classification of localizing $\otimes$-ideals and determines the Balmer spectrum in terms of $\text{Spec}(R)$. In this sense, BIK-stratification is both stronger and weaker than our notion of stratification.
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Thanks for listening!
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